Methods of Mathematical Physics

Preface:

I thank Allah Almighty who gave me a chance to serve the students in the field of Mathematics. I also thank the Holy Prophet Hazrat Muhammad (PBUH), who is forever a torch of guidance for humanity, without his goodwill nothing can be achieved. I am very grateful to my friends who encouraged me during the preparation of these notes. Further, I am thankful to my respected teacher **Prof. Dr.Azizullah Awan** for his kind support in this regard.

I feel myself lucky enough that my notes are popular all over Pakistan. Suggestions, comments, and indications of errors for the improvement will be highly appreciated.

I pray that my efforts will help the students in achieving their future objectives.

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Notes

M.Sc Part II (PU)

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Section II:

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CHAP 6:

Laplace Transformation.

Algebraic Equation.

An equation free from the derivative is called algebraic equation.

Laplace Transform.

Let $f(t)$ be a real-valued piecewise continuous function of t defined over the interval $[0, \infty)$, then the Laplace transform of $f(t)$ is a function $F(s)$ of another variable s defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$ and denoted by $L\{f(t)\}$.

Here e^{-st} is called the kernel of Laplace transformation.

Remark.

① Every continuous function is a piecewise continuous.

② Every function may not possess its Laplace transform.

Theorem.

The Laplace transformation Operator L is a linear Operator.

Proof.

Let $f(t) = c_1 f_1(t) + c_2 f_2(t)$

Then $L\{f(t)\} = L\{c_1 f_1(t) + c_2 f_2(t)\}$.

$$= \int_0^\infty e^{-st} \{c_1 f_1(t) + c_2 f_2(t)\}\, dt$$

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$$= C_1 \int_0^\infty e^{-st} f_1(t)\,dt + C_2 \int_0^\infty e^{-st} f_2(t)\,dt$$

$$= C_1 L\{f_1(t)\} + C_2 L\{f_2(t)\}$$

which shows that operator L is linear operator.

Exponential Order.

A function $f(t)$ is said to be of exponential order c if

$$|f(t)| \leq M e^{ct}, \text{ for } t > t_0.$$

where M and t_0 are +ve constants and c is also a constant.

Function of class A.

A function which is piecewise continuous and of exponential order is said to be a function of class A.

Theorem.

Sufficient condition for the existence of Laplace transform of a function is that it should be piecewise continuous and of exponential order.

Proof.

Let $f(t)$ be piecewise continuous in the interval $[0, T]$ and be of exponential order c.

Then there exist +ve real numbers M and t_0 such that

$$|f(t)| \leq M e^{ct}, \text{ for } t > t_0.$$

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$\Rightarrow |e^{-st} f(t)| \leq M e^{ct} \cdot e^{-st}$, for $t > t_0$.

$\Rightarrow |e^{-st} f(t)| \leq M e^{-(s-c)t}$, for $t > t_0$. ——— (1)

Now

$$|F(s)| = |L\{f(t)\}| = \left| \int_0^\infty e^{-st} f(t)\, dt \right|$$

$$\leq \int_0^\infty |e^{-st} f(t)|\, dt$$

$$\leq \int_0^\infty M e^{-(s-c)t}\, dt \quad \because \text{using } (1)$$

$$\Rightarrow F(s) \leq M \int_0^\infty e^{-(s-c)t}\, dt$$

$$= M \lim_{h \to \infty} \int_0^h e^{-(s-c)t}\, dt$$

$$= M \lim_{h \to \infty} \left| \frac{-e^{-(s-c)t}}{s-c} \right|_0^h$$

$$= M \lim_{h \to \infty} \left| \frac{-e^{-(s-c)h}}{s-c} + \frac{1}{s-c} \right|$$

$$= M |0 + 1/s-c|$$

$$= \frac{M}{|s-c|}$$

$$\Rightarrow F(s) \leq \frac{M}{|s-c|}, \text{ provided } s > c.$$

Thus piecewise continuity and exponential order are sufficient conditions for a function to have a Laplace transform.

But these are not necessary conditions.

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Note.

$L\{f(t)\} = F(s)$

then $\lim_{s\to\infty} F(s) = 0$ must satisfy.

First Shifting or Translation Properly. 2010,

Let $F(s) = L\{f(t)\}$ exist for $s > b$. For any constant 'a' such that $s > a+b$, $L\{e^{at} f(t)\}$ exists and $L\{e^{at} f(t)\} = F(s-a)$.

Proof.

By definition

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

so $$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt.$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt.$$

Put $s-a = s_1$ then

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-s_1 t} f(t)\,dt = F(s_1)$$

$$\Rightarrow L\{e^{at} f(t)\} = F(s-a) \quad \because s_1 = s-a.$$

Note.

The first shifting properly used to calculate Laplace transform of products of functions of the form $e^{at} f(t)$.

Prove that $L\{f(at)\} = \frac{1}{a} L\{f(t)\}$, $a > 0$

Proof.

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By definition

$$\mathcal{L}\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt \quad , \quad a>0. \qquad \text{——} (1)$$

Put $at = t_1$ then $t = \frac{1}{a}t_1 \Rightarrow dt = \frac{1}{a}dt_1$

so (1) $\Rightarrow \mathcal{L}\{f(t_1)\} = \int_0^\infty e^{-\frac{s}{a}t_1} f(t_1)\cdot\frac{1}{a}dt_1$

$$= \frac{1}{a}\int_0^\infty e^{-(\frac{s}{a})t_1} f(t_1)\,dt_1$$

Again put $s/a = s_1$ then

$$\mathcal{L}\{f(t_1)\} = \frac{1}{a}\int_0^\infty e^{-s_1 t_1} f(t_1)\,dt_1 = \frac{1}{a}\int_0^\infty e^{-s_1 t} f(t)\,dt$$

Def. integral میں variable تبدیل کرنے سے فرق نہیں پڑتا

$$= \frac{1}{a}\mathcal{L}\{f(t_1)\}$$

$$\Rightarrow \mathcal{L}\{f(at)\} = \frac{1}{a}\mathcal{L}\{f(t)\} \quad \text{or} \quad F(as) = \frac{1}{a}F(s).$$

The Differentiation Formula.

Let f be continuous on $[0,\infty)$ and of exponential order c. Let f' be piecewise continuous on every finite closed interval $0 \le t \le b$. Then $\mathcal{L}\{f'(t)\}$ exists for $s>c$ and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0).$$

Proof.

By definition

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty e^{-st}(-s) f(t)\,dt$$

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$$= 0 - f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + sL\{f(t)\}$$

$$= sF(s) - f(0).$$

Similarly

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

and continuing in this way, we get

$$L\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

Laplace Transform of the integral of a function.

If f is piecewise continuous and is of exponential order c, then

$$L\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}L\{f(t)\} = \frac{F(s)}{s}$$

Proof.

Let $g(t) = \int_0^t f(\tau)\,d\tau$, then by fundamental theorem of calculus

$$g'(t) = f(t)$$

$$\Rightarrow L\{g'(t)\} = sL\{g(t)\} - g(0)$$

But $$g(0) = \int_0^0 f(\tau)\,d\tau = 0$$

Hence $$L\{g'(t)\} = sL\{g(t)\}.$$

$$\Rightarrow \frac{1}{s}L\{g'(t)\} = L\{g(t)\}.$$

$$\Rightarrow L\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}L\{f(t)\} = \frac{F(s)}{s}$$

Inverse Laplace Transform.

If $F(s)$ is the Laplace transform of $f(t)$, then $f(t)$ is called the inverse Laplace transform of

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$F(s)$. It is denoted by $L^{-1}\{F(s)\} = f(t)$.

Example 1.

Calculate the Laplace transform of t^α, where α is any real number.

Solution.

By definition

$$L\{t^\alpha\} = \int_0^\infty e^{-st} t^\alpha \, dt$$

Put $st = u \Rightarrow dt = \frac{du}{s}$

then

$$L\{t^\alpha\} = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^\alpha \frac{du}{s} \qquad \because t = u/s$$

$$= \int_0^\infty e^{-u} \frac{u^\alpha}{s^\alpha} \frac{du}{s} = \int_0^\infty \frac{e^{-u} u^\alpha}{s^{\alpha+1}} du.$$

$$\Rightarrow L\{t^\alpha\} = \frac{1}{s^{\alpha+1}} \int_0^\infty e^{-u} u^\alpha du \qquad \text{——— (1)}$$

Since $\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} du$, gamma function,

Therefore, by putting $x = \alpha + 1$, we get

$$\Gamma(\alpha+1) = \int_0^\infty e^{-u} u^\alpha du$$

$$(1) \Rightarrow L\{t^\alpha\} = \frac{1}{s^{\alpha+1}} \Gamma(\alpha+1)$$

$$\text{or} \quad \boxed{L\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}} = \frac{\alpha!}{s^{\alpha+1}}$$

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Table of Some Laplace Transforms

$f(t)$	$L\{f(t)\}$	$f(t)$	$L\{f(t)\}$
1	$1/s$, $s>0$	t^n	$\frac{n!}{s^{n+1}}$
t^α, $\alpha>-1$	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$	e^{at}	$\frac{1}{s-a}$, $s>a$
te^{at}	$\frac{1}{(s-a)^2}$	$\sin at$	$\frac{a}{s^2+a^2}$, $s>0$
$\cos at$	$\frac{s}{s^2+a^2}$, $s>0$	$\sinh at$	$\frac{a}{s^2-a^2}$, $s>\lvert a\rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$, $s>\lvert a\rvert$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, $s>a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$	$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$e^{at}f(t)$	$F(s-a)$, $s>a$		

Example 2.

Find the inverse Laplace transforms of the following functions.

(i) $\frac{1}{s^2+2s}$ (ii) $\frac{s}{s^2+2s}$ (iii) $\frac{s+4}{s^2+3s+2}$

Solution.

(i)

$$\frac{1}{s^2+2s} = \frac{1}{s^2+2s+1-1} \neq \frac{1}{(s+1)^2-1} \neq \frac{1}{s(s+2)}$$

$$\Rightarrow \frac{1}{s^2+2s} = \frac{A}{s} + \frac{B}{s+2}$$ by Partial fraction.

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$\Rightarrow 1 = A(s+2) + B(s)$ ——— ①

Put $s=0$ in ①, we get,

$1 = A(2) \Rightarrow A = 1/2$

Put $s=-2$ in ①, we get

$1 = B(-2) \Rightarrow B = -1/2$

$① \Rightarrow \frac{1}{s^2+2s} = \frac{1}{2s} + (-\frac{1}{2})\frac{1}{s+2}$

$\Rightarrow \frac{1}{s^2+2s} = \frac{1}{2s} - \frac{1}{2(s+2)}$

Taking L^{-1} on both sides, we get.

$L^{-1}\left\{\frac{1}{s^2+2s}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s+2}\right\}.$

$= \frac{1}{2}(1) - \frac{1}{2}e^{-2t} = \frac{1}{2}(1-e^{-2t})$

$= \frac{1}{2}(e^{t}-e^{-t})e^{-t}$

$\Rightarrow L^{-1}\left\{\frac{1}{s^2+2s}\right\} = \sinh t\, e^{-t}.$

(ii)

$\frac{s}{s^2+2s} = \frac{s}{s(s+2)} = \frac{1}{s+2}$

Taking L^{-1} on both sides, we get

$L^{-1}\left\{\frac{s}{s^2+2s}\right\} = L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$

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(iii)

$$\frac{s+4}{s^2+3s+2} = \frac{s+4}{s^2+2s+s+2} = \frac{s+4}{s(s+2)+1(s+2)}$$

$$\Rightarrow \frac{s+4}{s^2+3s+2} = \frac{s+4}{(s+2)(s+1)}$$

$$= \frac{A}{s+1} + \frac{B}{s+2} \quad \text{by P. fraction}$$

$$\Rightarrow s+4 = A(s+2) + B(s+1) \quad \text{---------} \quad (1)$$

Put $s = -2$ then (1) implies

$$2 = B(-1) \Rightarrow \boxed{B = -2}$$

Put $s = -1$ then

$$(1) \Rightarrow 3 = A(1) \Rightarrow \boxed{A = 3}$$

So. $$\frac{s+4}{s^2+3s+2} = \frac{3}{s+1} - \frac{2}{s+2}.$$

Taking L^{-1} on both sides, we have,

$$L^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} = L^{-1}\left\{\frac{3}{s+1}\right\} - L^{-1}\left\{\frac{2}{s+2}\right\}$$

$$= 3L^{-1}\left\{\frac{1}{s+1}\right\} - 2L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= 3e^{-t} - 2e^{-2t}.$$

Example 3.

Solve the initial value problemes.

(i) $u' - 2u = 0$, $u(0) = 1$

(ii) $u'' + 4u' + 3u = 0$, $u(0) = 1$, $u'(0) = 0$.

Solution.

(i)

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Given that $u' - 2u = 0$.

Taking Laplace on both sides, we get,

$$L\{u' - 2u\} = 0$$

$$\Rightarrow L\{u'\} - 2L\{u\} = 0.$$

$$\Rightarrow sU(s) - u(0) - 2U(s) = 0$$ by formula of diff.

Applying given condition $u(0) = 1$

$$sU(s) - 1 - 2U(s) = 0$$

$$\Rightarrow U(s)(s-2) = 1$$

$$\Rightarrow U(s) = 1/s-2$$

Taking L^{-1} on both sides, we get,

$$L^{-1}\{U(s)\} = L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$\Rightarrow u(s) = e^{2t}.$$

(ii)

Given that $u'' + 4u' + 3u = 0$.

Taking Laplace on both sides, we get,

$$L\{u'' + 4u' + 3u\} = 0$$

$$\Rightarrow L\{u''\} + 4L\{u'\} + 3L\{u\} = 0$$

$$\Rightarrow s^2U(s) - su(0) - s^0u'(0) + 4(sU(s) - u(0)) + 3U(s) = 0$$

$$\Rightarrow s^2U(s) - su(0) - u'(0) + 4sU(s) - 4u(0) + 3U(s) = 0$$

$$\Rightarrow s^2U(s) + 4sU(s) + 3U(s) - su(0) - 4u(0) - u'(0) = 0$$

Applying given conditions, we get,

$$s^2U(s) + 4sU(s) + 3U(s) - s - 4 - 0 = 0$$

$$\Rightarrow U(s)(s^2 + 4s + 3) = s + 4$$

$$\Rightarrow U(s) = \frac{s+4}{s^2+4s+3}$$

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Taking L^{-1} on both sides, we get

$$L^{-1}\{U(s)\} = L^{-1}\left\{\frac{s+4}{s^2+4s+3}\right\}$$

$$\Rightarrow U(s) = L^{-1}\left\{\frac{s+4}{s^2+4s+3}\right\} \quad \text{---------- (1)}$$

Note that

$$\frac{s+4}{s^2+4s+3} = \frac{s+4}{s^2+3s+s+3} = \frac{s+4}{(s+3)(s+1)}$$

$$\Rightarrow \frac{s+4}{s^2+4s+3} = \frac{A}{s+3} + \frac{B}{s+1}$$ by partial fraction.

$$\Rightarrow s+4 = A(s+1) + B(s+3).$$

Put $s=-1$ then

$$3 = B(2) \Rightarrow \boxed{B = 3/2}$$

Put $s=-3$ then

$$1 = A(-2) \Rightarrow \boxed{A = -1/2}$$

So $$\frac{s+4}{s^2+4s+3} = \frac{-1}{2(s+3)} + \frac{3}{2}\cdot\frac{1}{s+1}$$

Thus equation (1) becomes

$$U(s) = -\frac{1}{2}L^{-1}\left\{\frac{1}{s+3}\right\} + \frac{3}{2}L^{-1}\left\{\frac{1}{s+1}\right\}.$$

$$= -\frac{1}{2}e^{-3t} + \frac{3}{2}e^{-t}.$$

Example 4.

Derive the Laplace transforms of $t^{1/2}$ and $t^{-1/2}$ from definition.

Solution.

Let $f(t) = t^{1/2}$.

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Taking Laplace on both sides, we get,

$$L\{f(t)\} = L\{t^{1/2}\}.$$

By definition

$$L\{f(t)\} = \int_{t=0}^{\infty} e^{-st} t^{1/2}\, dt$$

Put $st = s_1$ then $s\,dt = ds_1$

$$\Rightarrow L\{f(t)\} = \int_{s_1=0}^{\infty} e^{-s_1} \left(\frac{s_1}{s}\right)^{\frac{1}{2}} \frac{1}{s}\, ds_1$$

$$= \frac{1}{s^{3/2}} \int_0^{\infty} e^{-s_1} s_1^{1/2}\, ds_1 = 1/s^{3/2} \int_0^{\infty} e^{-s_1} s_1^{(1+\frac{1}{2})-1}\, ds_1$$

By gamma function $\Gamma(x) = \int_0^{\infty} e^{-u} u^{x-1}\, du$

$$L\{f(t)\} = \frac{1}{s^{3/2}} \Gamma\left(1+\frac{1}{2}\right) = \frac{1}{s^{3/2}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{s^{3/2}} \cdot \frac{1}{2} \sqrt{\pi} \qquad \because \Gamma(1/2) = \sqrt{\pi}.$$

$$= \frac{1}{2s} \cdot \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$\Rightarrow L\{f(t)\} = L\{t^{1/2}\} = \frac{1}{2s} \sqrt{\pi/s}.$$

Now let $f(t) = t^{-1/2}$

So $$L\{f(t)\} = L\{t^{-1/2}\} = \int_{s_1=0}^{\infty} e^{-s_1} \left(\frac{s_1}{s}\right)^{-1/2} \frac{1}{s}\, ds_1$$

$$= \frac{1}{s^{1/2}} \int_0^{\infty} e^{-s_1} s_1^{(1-\frac{1}{2})-1}\, ds_1$$

$$= \frac{1}{s^{1/2}} \Gamma\left(1-\frac{1}{2}\right) = \frac{1}{s^{1/2}} \Gamma(1/2) = \frac{1}{s^{1/2}} \sqrt{\pi}$$

$$\Rightarrow L\{t^{-1/2}\} = \sqrt{\frac{\pi}{s}}.$$

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Unit step Function.

The unit step function is also called Heaviside step function. It is denoted by $H(t-t_0)$ and is defined as

$$H(t-t_0) = \begin{cases} 1, & t \geq t_0 \\ 0, & t < t_0 \end{cases}$$

It is clear that the step function is discontinuous at $t=t_0$.

Convolution. 2016,

Let $f(t)$ and $g(t)$ be piecewise continuous functions over the interval $[0,\infty)$, then their convolution is a function of t denoted by $(f*g)(t)$ and defined by

$$(f*g)(t) = \int_{\tau=0}^{t} f(\tau)\, g(t-\tau)\, d\tau$$

From this definition prove the following results.

(i) $f*g = g*f$.

(ii) $f*(g+h) = (f*g) + (f*h)$.

(iii) $f*(g*h) = (f*g)*h$. see page (39)

Proof.

(i)

$$L.H.S = f \underset{t}{*} g$$

$$= \int_{\tau=0} f(\tau)\, g(t-\tau)\, d\tau \qquad \text{by defi.}$$

Put $t-\tau = t_1 \Rightarrow 0 - d\tau = dt_1$

$\Rightarrow d\tau = -dt_1$

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$\tau = 0 \Rightarrow t_1 = t$, $\quad \tau = t \Rightarrow t_1 = 0$.

then $f * g = \int_{t_1=t}^{0} f(t-t_1)\, g(t_1)\, (-dt_1)$.

$= \int_{t_1=0}^{t} f(t-t_1)\, g(t_1)\, dt_1 = g * f = R.H.S.$

(ii).

$L.H.S = f*(g+h) = \int_{\tau=0}^{t} f(\tau)(g+h)(t-\tau)\, d\tau$

$= \int_{0}^{t} f(\tau)\{g(t-\tau) + h(t-\tau)\} d\tau$.

$= \int_{0}^{t} f(\tau)\, g(t-\tau)\, d\tau + \int_{0}^{t} f(\tau) h(t-\tau) d\tau$

$= (f*g) + (f*h) = R.H.S.$

(iii).

This part cannot be solved easily but try try again.

Special Convolution (Faltung) Theorem. 2013, 2015, 2016

Let $f(t)$ and $g(t)$ be piecewise continuous functions over the interval $[0, \infty)$ and of exponential order and $L\{f(t)\} = F(s)$, $L\{g(t)\} = G(s)$. Then

$$L\{(f*g)\} = F(s)\, G(s).$$

Proof.

By definition

$$L\{f*g\} = \int_{t=0}^{\infty} e^{-st} (f*g)\, dt \quad \text{——} (1)$$

Since we know that

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$$f*g = \int_{\tau=0}^{\infty} H(t-\tau)\, f(\tau)\, g(t-\tau)\, d\tau$$

where H is the step function.

Therefore ① becomes

$$L\{f*g\} = \int_{t=0}^{\infty} e^{-st} \int_{\tau=0}^{\infty} H(t-\tau) f(\tau) g(t-\tau)\, d\tau\, dt$$

By changing the order of integration, we get,

$$L\{f*g\} = \int_{\tau=0}^{\infty} f(\tau) \left[\int_{t=0}^{\infty} e^{-st} H(t-\tau) g(t-\tau)\, dt \right] d\tau.$$

Put $t-\tau = t_1 \Rightarrow dt - d\tau = dt_1 \Rightarrow dt = dt_1$

$t=0 \Rightarrow t_1 = -\tau$; $t \to \infty$, $t_1 \to \infty$.

then $$L\{f*g\} = \int_{\tau=0}^{\infty} f(\tau) \left[\int_{t_1=-\tau}^{\infty} e^{-s(t_1+\tau)} H(t_1) g(t_1) dt_1 \right] d\tau.$$

$$= \int_{\tau=0}^{\infty} f(\tau) \left[\int_{t_1=-\tau}^{\infty} e^{-st_1} e^{-s\tau} H(t_1) g(t_1) dt_1 \right] d\tau$$

$$= \int_{\tau=0}^{\infty} e^{-s\tau} f(\tau) \left[\int_{t_1=-\tau}^{\infty} e^{-st_1} H(t_1) g(t_1) dt_1 \right] d\tau.$$

$$= \int_{\tau=0}^{\infty} e^{-s\tau} f(\tau) \left[\int_{t_1=-\tau}^{0} e^{-st_1} H(t_1) g(t_1) dt_1 + \int_{0}^{\infty} e^{-st_1} H(t_1) g(t_1) dt_1 \right] d\tau.$$

Since $$H(t-\tau) = H(t_1) = \begin{cases} 0, & t_1 < 0 \\ 1, & t_1 \geq 0 \end{cases}$$

So $$\int_{t_1=-\tau}^{0} e^{-st_1} H(t_1) g(t_1)\, dt_1 = 0.$$

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Thus $L\{f*g\} = \int_{t=0}^{\infty} e^{-st} f(t) \left\{ \int_{t_1=0}^{\infty} e^{-st_1} H(t_1)\, g(t_1)\, dt_1 \right\} dt$.

Where $H(t_1) = 1$.

$$\Rightarrow L\{f*g\} = \int_{t=0}^{\infty} f(t) e^{-st} \left\{ \int_{t_1=0}^{\infty} e^{-st_1} g(t_1)\, dt_1 \right\} dt$$

$$= \int_{t=0}^{\infty} e^{-st} f(t)\, dt \cdot \int_{t_1=0}^{\infty} e^{-st_1} g(t_1)\, dt_1$$

As we can change variable in definite integral so

$$L\{f*g\} = \int_{t=0}^{\infty} e^{-st} f(t)\, dt \cdot \int_{t=0}^{\infty} e^{-st} g(t)\, dt.$$

$$= L\{f(t)\} \cdot L\{g(t)\} = F(s) \cdot G(s).$$

$$\Rightarrow L\{f*g\} = F(s)\, G(s).$$ as required.

Example 1.

Using the convolution theorem, calculate the inverse Laplace transform of the function

(i) $\dfrac{3}{s^2(s^2+9)}$ (ii) 2017 $\dfrac{s}{(s^2+9)^2}$

Special $\left\{ \dfrac{s}{(s^2+a)^2} \right\}$

Solution.

(i)

Let $F(s) = 1/s^2$, $G(s) = \dfrac{3}{s^2+9}$

Then $L^{-1}\{F(s)\} = L^{-1}\{1/s^2\}$; $L^{-1}\{G(s)\} = L^{-1}\left\{\dfrac{3}{s^2+9}\right\}$

$\Rightarrow f(t) = t$; $g(t) = \sin 3t$.

Using Convolution theorem.

$$L^{-1}\left\{\frac{3}{s^2(s^2+9)}\right\} = L^{-1}\{F(s)G(s)\} = (f*g)(t)$$

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$$= \int_0^t f(\bar{t})\, g(t-\bar{t})\, d\bar{t}$$

$$= \int_0^t \bar{t} \sin 3(t-\bar{t})\, d\bar{t}$$

$$= -\bar{t}\, \frac{\cos 3(t-\bar{t})}{-3}\Big|_0^t + \int_0^t \frac{\cos 3(t-\bar{t})}{-3}\, d\bar{t}$$

$$= +\frac{t}{3}(1) - 0 - \frac{1}{3}\left\{ \frac{\sin 3(t-\bar{t})}{-3} \right\}_0^t$$

$$= \frac{t}{3} + \frac{1}{9}\{0 - \sin 3t\}$$

$$= \frac{1}{9}\{3t - \sin 3t\}$$

(iii) $H(s) = F(s)\, G(s)$

Let $F(s) = \frac{s}{s^2+9}$, $G(s) = \frac{1}{s^2+9} = \frac{1}{3}\frac{3}{s^2+9}$

$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s}{s^2+9}\right\}$; $L^{-1}\{G(s)\} = \frac{1}{3} L^{-1}\left\{\frac{3}{s^2+9}\right\}$

$\Rightarrow f(t) = \cos 3t$; $g(t) = \frac{1}{3}\sin 3t$

$$(f*g)(t) = \int_{\bar{t}=0}^{t} f(\bar{t})\, g(t-\bar{t})\, d\bar{t}$$

$$= \frac{1}{3}\int_0^t \cos 3\bar{t} \sin 3(t-\bar{t})\, d\bar{t}$$

$$= \frac{1}{3}\int_0^t \cos 3\bar{t}\, (\sin 3t \cos 3\bar{t} - \cos 3t \sin 3\bar{t})\, d\bar{t}$$

$$= \frac{1}{3}\int_0^t (\sin 3t \cos 3\bar{t} \cos 3\bar{t} - \cos 3t \cos 3\bar{t} \sin 3\bar{t})\, dt$$

$$= \frac{1}{3}\left[\sin 3t \cdot \frac{1}{2}\int_{\bar{t}=0}^{t} 2\cos 3\bar{t} \cos 3\bar{t}\, d\bar{t} - \frac{1}{2}\cos 3t \times \int_{\bar{t}=0}^{t} 2\cos 3\bar{t} \sin 3\bar{t}\, d\bar{t}\right].$$

$$= \frac{1}{6}\left[\sin 3t \int_0^t \{\cos(3\bar{t}+3\bar{t}) + \cos(0)\}\, d\bar{t} -\right.$$

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$\frac{1}{6}\cos 3t \int_0^t \{\sin(3\bar{t}+3t) - \sin(0)\} d\bar{t}$]

$= \frac{1}{6}\left[\sin 3t \int_0^t (\cos 6\bar{t} + 1)\, d\bar{t} - \cos 3t \int_0^t \sin 6\bar{t}\, d\bar{t}\right]$.

$= \frac{1}{6}\sin 3t \left\{\frac{\sin 6\bar{t}}{6} + \bar{t}\right\}_0^t - \frac{1}{6}\cos 3t \left\{-\frac{\cos 6\bar{t}}{6}\right\}_0^t$

$= \frac{1}{6}\sin 3t \left\{\frac{1}{6}\sin 6t + t\right\} - \frac{1}{6}\cos 3t \left\{-\frac{1}{6}\cos 6t + \frac{1}{6}\right\}$

$= \frac{1}{6} t \sin 3t + \frac{1}{36}\sin 3t \sin 6t + \frac{1}{36}\cos 3t \cos 6t - \frac{1}{36}\cos 3t$

$= \frac{1}{6} t \sin 3t - \frac{1}{36}\cos 3t + \frac{1}{36}\{\sin 3t \sin 6t + \cos 3t \cos 6t\}$.

$= \frac{1}{6} t \sin 3t - \frac{1}{36}\cos 3t + \frac{1}{36}\{\cos(6t - 3t)\}$.

$= \frac{1}{6} t \sin 3t - \frac{1}{36}\cos 3t + \frac{1}{36}\cos 3t$

$= \frac{1}{6} t \sin 3t$.

Example 2.

Use convolution theorem to calculate the Laplace transform of

(i) $f(t) = \int_0^t (t-\beta)^3 e^{\beta} \sin\beta\, d\beta$

(ii) $f(t) = \int_0^t (t-\beta) \sin 3\beta\, d\beta$

(iii) $f(t) = \int_0^t e^{-(t-\beta)} \sin\beta\, d\beta$

Solution.

(i)

Given that $f(t) = \int_0^t (t-\beta)^3 e^{\beta} \sin\beta\, d\beta$.

So by definition of convolution,

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$$f(t) * g(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

so we have

$$f(t) = t^3 * e^t \sin t$$

Taking Laplace on both sides, we get,

$$L\{f(t)\} = L\{e^t \sin t\}\, L\{t^3\}.$$

$$= \frac{3!}{s^4} \cdot \frac{1}{(s-1)^2+(1)^2}$$ by first shifting prop.

$$= \frac{6}{s^4(s^2+1-2s+1)} = \frac{6}{s^4(s^2-2s+2)}.$$

(ii)

Given that $f(t) = \int_0^t (t-\beta) \sin 3\beta\, d\beta$.

By definition of convolution,

$$f * g = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

so we have

$$f(t) = t * \sin 3t$$

Taking Laplace on both sides, we get,

$$L\{f(t)\} = L\{t\}\, L\{\sin 3t\}$$

$$= \frac{1}{s^2} \cdot \frac{3}{s^2+9} = \frac{3}{s^2(s^2+9)}.$$

(iii)

Given that $f(t) = \int_0^t e^{-(t-\beta)} \sin\beta\, d\beta$

By definition of convolution

$$f * g = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

So $f(t) = e^{-t} * \sin t$.

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taking Laplace on both sides

$$L\{f(t)\} = L\{e^{-t}\}\, L\{\sin t\}$$

$$= \frac{1}{s+1} \cdot \frac{1}{s^2+1} = \frac{1}{(s+1)(s^2+1)}$$

Exercise:

Question 1.

Show that $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_{\tau=0}^{t} f(\tau)\, d\tau$.

Solution.

By Convolution theorem,

$$L\{f * g\} = F(s)\, G(s).$$

$$\Rightarrow L^{-1}\{F(s)G(s)\} = f * g = \int_{\tau=0}^{t} f(\tau)\, g(t-\tau)\, d\tau$$

$$\Rightarrow L^{-1}\{F(s)G(s)\} = \int_{0}^{t} f(\tau)\, g(t-\tau)\, d\tau \quad \text{——} (1)$$

Let $G(s) = \frac{1}{s}$ then $L^{-1}\{G(s)\} = L^{-1}\{1/s\}$

$$\Rightarrow g(t) = 1$$

$$\Rightarrow g(t-\tau) = 1.$$

Thus (1) $\Rightarrow L^{-1}\left\{F(s)\frac{1}{s}\right\} = \int_{0}^{t} f(\tau) \cdot 1 \cdot d\tau$

$$\Rightarrow L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_{0}^{t} f(\tau)\, d\tau.$$

Question 2:

Show that $L^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_{\tau=0}^{t}\left\{\int_{\lambda=0}^{\tau} f(\lambda)\, d\lambda\right\} d\tau$.

Solution.

$$L.H.S = L^{-1}\left\{\frac{F(s)}{s^2}\right\} = L^{-1}\left\{\frac{F(s)}{s} \cdot \frac{1}{s}\right\}$$

$$= \left(\int_{0}^{t} f(\lambda)\, d\lambda\right) * 1 \quad \text{——} (1) \text{ by Q1.}$$

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$\Rightarrow f(t) = \int_0^t f(\lambda)\,d\lambda$; $g(t) = 1 \Rightarrow g(t-\tau) = 1$.

By definition of convolution

$$f * g = \int_{\tau=0}^{t} f(\tau)\, g(t-\tau)\, d\tau.$$

So $\left(\int_0^t f(\lambda)\,d\lambda\right) * 1 = \int_{\tau=0}^{t} \left(\int_{\lambda=0}^{\tau} f(\lambda)\,d\lambda\right) \cdot 1 \, d\tau$.

$\Rightarrow \left(\int_0^t f(\lambda)\,d\lambda\right) * 1 = \int_0^t \left(\int_0^\tau f(\lambda)\,d\lambda\right) d\tau$.

Thus ① $\Rightarrow L^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \left(\int_0^\tau f(\lambda)\,d\lambda\right) d\tau$.

Q3:

Solve the DEs by Laplace transform method.

Special (a) $y''(t) + k^2 y(t) = f(t)$

(b) $y''(t) - 2ky'(t) + k^2 y(t) = f(t)$

Special (c) $y''(t) + \lambda y'(t) + k^2 y(t) = f(t)$.

Solution.

(a)

$L\{y''(t)\} + k^2 L\{y(t)\} = L\{f(t)\}$

$\Rightarrow s^2 Y(s) - s y(0) - y'(0) + k^2 Y(s) = F(s)$

$\Rightarrow (s^2 + k^2) Y(s) = F(s) + s y(0) + y'(0)$

Let $y(0) = C_1$ (say)

$y'(0) = C_2$

$\Rightarrow (s^2 + k^2) Y(s) = F(s) + C_1 s + C_2$

$\Rightarrow Y(s) = \frac{F(s) + C_1 s + C_2}{s^2 + k^2}$

Taking L^{-1} on both sides, we get,

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$$L^{-1}\{y(s)\} = L^{-1}\left\{\frac{F(s)}{s^2+k^2}\right\} + C_1 L^{-1}\left\{\frac{s}{s^2+k^2}\right\} + C_2 L^{-1}\left\{\frac{1}{s^2+k^2}\right\}$$

$$= \frac{1}{k} L^{-1}\left\{F(s)\cdot\frac{k}{s^2+k^2}\right\} + C_1 \cos kt + \frac{1}{k} C_2 L^{-1}\left\{\frac{k}{s^2+k^2}\right\}$$

$$= \frac{1}{k} L^{-1}\left\{F(s)\cdot\frac{k}{s^2+k^2}\right\} + C_1 \cos kt + \frac{C_2}{k}\sin kt \quad \text{---(1)}$$

Note that

$$L^{-1}\left\{F(s)\cdot\frac{k}{s^2+k^2}\right\} = f(t) * \sin kt = \int_0^t f(\bar{t})\sin k(t-\bar{t})\,d\bar{t}$$

(convolution)

$$\left(= \int_0^t f(\bar{t})\{\sin kt\cos k\bar{t} - \cos kt \sin k\bar{t}\}d\bar{t}\right.$$

$$= \int_0^t f(\bar{t})\sin kt\cos k\bar{t}\,d\bar{t} - \int_0^t f(\bar{t})\cos kt\sin k\bar{t}\,d\bar{t}$$

$$\left.\right)$$

Thus (1) becomes

$$L^{-1}\{y(s)\} = \frac{1}{k}\int_0^t f(\bar{t})\sin kt\cos k\bar{t}\,d\bar{t} - \frac{1}{k}\int_0^t f(\bar{t})\cos kt\sin k\bar{t}\,d\bar{t} + C_1\cos kt + \frac{C_2}{k}\sin kt.$$

$$\Rightarrow y(t) = \frac{1}{k}\sin kt\int_0^t f(\bar{t})\cos k\bar{t}\,d\bar{t} - \frac{1}{k}\cos kt\int_0^t f(\bar{t})\sin k\bar{t}\,d\bar{t} + C_1\cos kt + \frac{C_2}{k}\sin kt.$$

(b).

$$L\{y''(t)\} - 2kL\{y'(t)\} + k^2 L\{y(t)\} = L\{f(t)\}.$$

$$\Rightarrow s^2 y(s) - sy(0) - y'(0) - 2k\{sy(s) - y(0)\} + k^2 y(s) = F(s)$$

$$\Rightarrow s^2 y(s) - sy(0) - y'(0) - 2ksy(s) + 2ky(0) + k^2 y(s) = F(s)$$

$$\Rightarrow y(s)(s^2 - 2ks + k^2) = F(s) + sy(0) + y'(0) - 2ky(0)$$

$$y(s) = \frac{F(s) + (s-2k)C_1 + C_2}{(s-k)^2}$$

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$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{F(s)}{(s-k)^2}\right\} + C_1 L^{-1}\left\{\frac{1}{s-k}\right\} - C_1 k L^{-1}\left\{\frac{1}{(s-k)^2}\right\} + C_2 L^{-1}\left\{\frac{1}{(s-k)^2}\right\}.$$

$$\Rightarrow y(t) = L^{-1}\left\{\frac{F(s)}{(s-k)^2}\right\} + C_1 e^{kt} - C_1 k e^{kt} t + C_2 t e^{kt} \quad \text{---(1)}$$

Consider

$$L^{-1}\left\{\frac{F(s)}{(s-k)^2}\right\} = f(t) * e^{kt} t$$

$$(1) \Rightarrow y(t) = te^{kt} * f(t) + C_1 e^{kt} - C_1 k t e^{kt} + C_2 t e^{kt}$$

(C)

$$L\{y''(t)\} + \lambda L\{y'(t)\} + k^2 L\{y(t)\} = L\{f(t)\}.$$

$$s^2 Y(s) - s y(0) - y'(0) + \lambda\{s Y(s) - y(0)\} + k^2 Y(s) = F(s).$$

$$\Rightarrow s^2 Y(s) - s y(0) - y'(0) + \lambda s Y(s) - \lambda y(0) + k^2 Y(s) = F(s)$$

$$(s^2 + \lambda s + k^2) Y(s) = F(s) + s y(0) + y'(0) + \lambda y(0)$$

$$\Rightarrow Y(s) = \frac{F(s) + (s+\lambda) y(0) + y'(0)}{s^2 + \lambda s + k^2}$$

Let $y(0) = C_1$ (say).

$y'(0) = C_2$

Then $Y(s) = \dfrac{F(s) + (s+\lambda) C_1 + C_2}{s^2 + \lambda s + k^2}$

$$= \frac{F(s) + (s+\lambda) C_1 + C_2}{s^2 + 2\frac{\lambda}{2} s + k^2 + \frac{\lambda^2}{4} - \frac{\lambda^2}{4}}$$

$$= \frac{F(s) + (s+\lambda) C_1 + C_2}{(s+\frac{\lambda}{2})^2 + k^2 - \lambda^2/4}$$

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Question 4.

Solve the problem

$$y'' + \omega^2 y = f(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

and discuss the case when

$$f(t) = \begin{cases} f_0, & t_0 < t < t_1 \\ 0, & \text{for other values} \end{cases}.$$

Solution.

Given that $y'' + \omega^2 y = f(t)$.

Taking Laplace transform on both sides, we have,

$$L\{y''\} + \omega^2 L\{y\} = L\{f(t)\}.$$

$$\Rightarrow s^2 Y(s) - s y(0) - y'(0) + \omega^2 Y(s) = F(s)$$

$$\Rightarrow (s^2 + \omega^2) Y(s) = F(s) + s y(0) + y'(0)$$

using $y(0) = y_0$ and $y'(0) = y_1$ to get

$$(s^2 + \omega^2) Y(s) = F(s) + s y_0 + y_1$$

$$\Rightarrow Y(s) = \frac{F(s) + s y_0 + y_1}{s^2 + \omega^2}$$

$$\Rightarrow Y(s) = \frac{1}{\omega} \cdot \frac{F(s)\,\omega}{s^2 + \omega^2} + \frac{s y_0}{s^2 + \omega^2} + \frac{1}{\omega} \frac{y_1 \omega}{s^2 + \omega^2}$$

Taking L^{-1}, we get

$$L^{-1}\{Y(s)\} = \frac{1}{\omega} L^{-1}\left\{F(s) \cdot \frac{\omega}{s^2 + \omega^2}\right\} + y_0 L^{-1}\left\{\frac{s}{s^2 + \omega^2}\right\}$$

$$+ \frac{1}{\omega} y_1 L^{-1}\left\{\frac{\omega}{s^2 + \omega^2}\right\}.$$

$$= \frac{1}{\omega} f(t) * \sin \omega t + y_0 \cos \omega t + \frac{y_1}{\omega} \sin \omega t$$

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$$\Rightarrow y(t) = y_0 \cos\omega t + \frac{y_1}{\omega}\sin\omega t + \frac{1}{\omega}\int_{t_0}^{t} f(\tau)\sin\omega(t-\tau)d\tau$$

$$= y_0 \cos\omega t + \frac{y_1}{\omega}\sin\omega t + \frac{1}{\omega}\int_{0}^{t_1} f(\tau)\sin\omega(t-\tau)d\tau$$

$$+ \frac{1}{\omega}\int_{t_0}^{t} f(\tau)\sin\omega(t-\tau)d\tau + \frac{1}{\omega}\int_{t_1}^{t} f(\tau)\sin\omega(t-\tau)d\tau.$$

Using given condition

$$f(t) = \begin{cases} f_0 & , t_0 < t < t_1 \\ 0 & \text{otherwise} \end{cases}, \text{ we get.}$$

$$y(t) = y_0 \cos\omega t + \frac{y_1}{\omega}\sin\omega t + \frac{1}{\omega}\int_{t_0}^{t_1} f(\tau)\sin\omega(t-\tau)d\tau.$$

Question 5:

Solve the inhomogeneous problem with zero initial conditions i.e. $u(0)=0$ and $u'(0)=0$.

(i) $u'' + au = 1$

Given that $u'' + au = 1$

Taking Laplace transform, we get,

$$L\{u''\} + aL\{u\} = L\{1\}.$$

$$\Rightarrow s^2U(s) - su(0) - u'(0) + aU(s) = 1/s$$

$$\Rightarrow (s^2+a)U(s) = \frac{1}{s} + su(0) + u'(0)$$

By given conditions $u(0)=0$ and $u'(0)=0$

$$(s^2+a)U(s) = 1/s$$

$$\Rightarrow U(s) = \frac{1}{s(s^2+a)} = \frac{1}{s(s^2+(\sqrt{a})^2)}$$

$$\Rightarrow U(s) = \frac{1}{\sqrt{a}}\left[\frac{1}{s}\cdot\frac{\sqrt{a}}{s^2+(\sqrt{a})^2}\right]$$

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Taking L^{-1} on both sides, we have.

$$L^{-1}\{U(s)\} = \frac{1}{\sqrt{a}} L^{-1}\left\{\frac{1}{s}\cdot\frac{\sqrt{a}}{s^2+(\sqrt{a})^2}\right\}$$

By convolution, theorem

$$u(t) = \frac{1}{\sqrt{a}}(1 * \sin\sqrt{a}t)$$

$$= \frac{1}{\sqrt{a}}\int_{\tau=0}^{t} \sin\sqrt{a}\tau \, d\tau \qquad \because f(\tau)=1,\ f(t-\tau)=1.$$

$$= \frac{1}{\sqrt{a}}\left[-\frac{\cos\sqrt{a}\tau}{\sqrt{a}}\right]_{\tau=0}^{t} = \frac{-1}{a}\left[\cos\sqrt{a}t - 1\right].$$

$$\Rightarrow u(t) = \frac{1}{a}(1-\cos\sqrt{a}t).$$

(ii) $u'' + u = t$

Taking Laplace transform, we get

$$L\{u''\} + L\{u\} = L\{t\}.$$

$$\Rightarrow s^2U(s) - su(0) - u'(0) + U(s) = 1/s^2$$

$$\Rightarrow (s^2+1)U(s) = 1/s^2 \qquad \text{using given condi.}$$

$$\Rightarrow U(s) = \frac{1}{s^2(s^2+1)}.$$

Taking L^{-1} on both sides, we get.

$$L^{-1}\{U(s)\} = L^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s^2+1}\right\}.$$

$$\Rightarrow u(t) = t * \sin t \qquad \text{by convolution.}$$

$$= \int_{\tau=0}^{t} \tau \sin(t-\tau)\, d\tau.$$

$$= \int_0^t \tau\left[\sin t\cos\tau - \cos t\sin\tau\right] d\tau.$$

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$$= \int_0^t (\bar{t}\sin t\cos\bar{t} - \bar{t}\cos t\sin\bar{t})\,d\bar{t}$$

$$= \int_0^t \bar{t}\sin t\cos\bar{t}\,d\bar{t} - \int_0^t \bar{t}\cos t\sin\bar{t}\,d\bar{t}.$$

$$= \sin t \int_{\bar{t}=0}^{t} \bar{t}\cos\bar{t}\,d\bar{t} - \cos t \int_{\bar{t}=0}^{t} \bar{t}\sin\bar{t}\,d\bar{t}.$$

Integrating by parts, we get,

$$= \sin t\{\bar{t}\sin\bar{t}\Big|_0^t - \int_0^t \sin\bar{t}\,d\bar{t}\} - \cos t\{-\bar{t}\cos\bar{t}\Big|_0^t + \int_0^t \cos\bar{t}\,d\bar{t}\}$$

$$= \sin t\{t\sin t + \cos\bar{t}\Big|_0^t\} - \cos t\{-t\cos t + \sin\bar{t}\Big|_0^t\}.$$

$$= \sin t\{t\sin t + \cos t - 1\} - \cos t\{-t\cos t + \sin t\}.$$

$$= t\sin^2 t + \sin t\cos t - \sin t + t\cos^2 t - \sin t\cos t$$

$$= t(\sin^2 t + \cos^2 t) - \sin t$$

$$= t - \sin t.$$

(iii) $u'' + 2u' = 1 - e^{-t}$. 2013/A.

Taking Laplace transform, we get,

$$L\{u''\} + 2L\{u'\} = L\{1\} - L\{e^{-t}\}.$$

$$\Rightarrow s^2U(s) - su(0) - u'(0) + 2\{sU(s) - u(0)\} = \frac{1}{s} - \frac{1}{s+1}$$

$$\Rightarrow s^2U(s) + 2sU(s) = \frac{1}{s} - \frac{1}{s+1}$$ by given cond.

$$\Rightarrow (s^2+2s)U(s) = \frac{1}{s} - \frac{1}{s+1}$$

$$\Rightarrow (s+2)U(s) = \frac{1}{s^2} - \frac{1}{s(s+1)}.$$

$$\Rightarrow U(s) = \frac{1}{s^2(s+2)} - \frac{1}{s(s+1)(s+2)}.$$

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Taking L^{-1} on both sides, we get

$$L^{-1}\{U(s)\} = L^{-1}\left\{\frac{1}{s^2(s+2)}\right\} - L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\}.$$

$$= L^{-1}\left\{\frac{1}{s^2(s+2)} - \frac{1}{s(s+1)(s+2)}\right\}.$$

$$= L^{-1}\left\{\frac{s+1-s}{s^2(s+1)(s+2)}\right\}.$$

$$\Rightarrow u(t) = L^{-1}\left\{\frac{1}{s^2(s+1)(s+2)}\right\} \quad \text{——— (1)}$$

Note that

$$\frac{1}{s^2(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2}$$ by partial fraction.

$$\Rightarrow 1 = As(s+1)(s+2) + B(s+1)(s+2) + C(s+2)s^2 + Ds^2(s+1).$$

Put $s = 0$ then

$1 = B(1)(2) \Rightarrow \boxed{B = 1/2}$

Put $s = -1$ then

$1 = C(1)(1) \Rightarrow \boxed{C = 1}$

Put $s = -2$ then

$1 = D(-2)^2(-2+1) \Rightarrow -4D = 1 \Rightarrow \boxed{D = -\frac{1}{4}}$

Now

$$1 = As(s^2+2s+s+2) + B(s^2+3s+2) + Cs^2(s+2) + Ds^2(s+1)$$

Comparing coefficients of s^3, we get

$$0 = A + C + D$$

$$\Rightarrow A = -C - D = -1 + \frac{1}{4} = \frac{-4+1}{4}$$

$$\Rightarrow \boxed{A = \frac{-3}{4}}$$

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So $\frac{1}{s^2(s+1)(s+2)} = -\frac{3}{4s} + \frac{1}{2s^2} + \frac{1}{s+1} - \frac{1}{4}\cdot\frac{1}{s+2}$

Taking L^{-1} on both sides, we get,

$$L^{-1}\left\{\frac{1}{s^2(s+1)(s+2)}\right\} = -\frac{3}{4}L^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{s^2}\right\} + L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{4}L^{-1}\left\{\frac{1}{s+2}\right\}.$$

$$= -\frac{3}{4}(1) + \frac{1}{2}t + e^{-t} - \frac{1}{4}e^{-2t}.$$

Thus equation ① becomes

$$u(t) = \frac{1}{2}t + e^{-t} - \frac{1}{4}e^{-2t} - \frac{3}{4}.$$

(iv) $u'' - u = 1$

Taking Laplace transform on both sides, we have,

$$L\{u''\} - L\{u\} = L\{1\}.$$

$\Rightarrow s^2U(s) - su(0) - u'(0) - U(s) = 1/s$

$\Rightarrow (s^2-1)U(s) = 1/s$ by given conditions

$\Rightarrow U(s) = \frac{1}{s(s^2-1)} = \frac{1}{s(s+1)(s-1)}$

Taking L^{-1}, we get

$$L^{-1}\{U(s)\} = L^{-1}\left\{\frac{1}{s(s+1)(s-1)}\right\}. \quad \text{——— ①}$$

Note that

$$\frac{1}{s(s+1)(s-1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-1}$$ by partial fraction

$\Rightarrow 1 = A(s^2-1) + Bs(s-1) + Cs(s+1)$

Put $s=0$ then

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$1 = (-A) \Rightarrow \boxed{A = -1}$

Put $s = -1$ then

$1 = B(-1)(-2) \Rightarrow \boxed{B = 1/2}$

Put $s = 1$ then

$1 = 2C \Rightarrow \boxed{C = 1/2}$

So

$$\frac{1}{s(s+1)(s-1)} = \frac{-1}{s} + \frac{1}{2(s+1)} + \frac{1}{2(s-1)}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s(s+1)(s-1)}\right\} = -L^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= -1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}$$

Thus equation (1) becomes

$$u(t) = \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} - 1$$

$$= \frac{1}{2}(e^{t} + e^{-t}) - 1$$

$$= \cosh t - 1.$$

(V)

$$u'' + 4u = \sin t$$

Taking Laplace transform on both sides, we get,

$$L\{u''\} + 4L\{u\} = L\{\sin t\}.$$

$$\Rightarrow s^2U(s) - su(0) - u'(0) + 4U(s) = \frac{1}{s^2+1}$$

$$\Rightarrow (s^2+4)U(s) = \frac{1}{s^2+1}$$ by given conditions

$$\Rightarrow U(s) = \frac{1}{(s^2+4)(s^2+1)}$$

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taking L^{-1} of both sides, we get,

$$L^{-1}\{U(s)\} = \frac{1}{2}L^{-1}\left\{\frac{2}{s^2+4}\cdot\frac{1}{s^2+1}\right\}$$

By convolution theorem.

$$u(t) = \frac{1}{2}\sin 2t * \sin t$$

$$= \frac{1}{2}\int_{\bar{t}=0}^{t} \sin 2\bar{t}\,\sin(t-\bar{t})\,d\bar{t}$$

$$= \frac{1}{2}\int_{0}^{t} \sin 2\bar{t}\,(\sin t\cos\bar{t} - \cos t\sin\bar{t})\,d\bar{t}$$

$$= \frac{1}{2}\sin t\int_{\bar{t}=0}^{t}\sin 2\bar{t}\cos\bar{t}\,d\bar{t} - \frac{1}{2}\cos t\int_{\bar{t}=0}^{t}\sin 2\bar{t}\sin\bar{t}\,d\bar{t}$$

$$= \frac{1}{4}\sin t\int_{0}^{t} 2\sin 2\bar{t}\cos\bar{t}\,d\bar{t} - \frac{1}{4}\cos t\int_{0}^{t} 2\sin 2\bar{t}\sin\bar{t}\,d\bar{t}$$

$$= \frac{1}{4}\sin t\int_{0}^{t}\{\sin(3\bar{t})+\sin(\bar{t})\}d\bar{t} + \frac{1}{4}\cos t\int_{0}^{t}\{\cos(3\bar{t})-\cos\bar{t}\}d\bar{t}$$

$$= \frac{1}{4}\sin t\left\{-\frac{\cos 3\bar{t}}{3}\Big|_0^t - \cos\bar{t}\Big|_0^t\right\} + \frac{1}{4}\cos t\left\{\frac{\sin 3\bar{t}}{3}\Big|_0^t - \sin\bar{t}\Big|_0^t\right\}$$

$$= \frac{1}{4}\sin t\left\{-\frac{1}{3}\cos 3t + \frac{1}{3} - \cos t + 1\right\} + \frac{1}{4}\cos t\left\{\frac{1}{3}\sin 3t - \sin t\right\}$$

$$= \frac{1}{12}\sin t\{-\cos 3t + 1 - 3\cos t + 3\} + \frac{1}{12}\cos t\{\sin 3t - 3\sin t\}$$

$$= \frac{1}{12}[-\sin t\cos 3t + \sin t - 3\sin t\cos t + 3\sin t + \cos t\sin 3t - 3\cos t\sin t]$$

end

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Question 6. 2015)

Solve the I.V. Problem

$y''(t) + t\,y'(t) - y(t) = 0$, $y(0)=0$, $y'(0)=1$.

Solution.

Given that

$$y''(t) + t\,y'(t) - y(t) = 0$$

Taking Laplace transformation, we get.

$$L\{y''(t)\} + L\{t\,y'(t)\} - L\{y(t)\} = 0.$$

$$\Rightarrow s^2 Y(s) - s\,y(0) - y'(0) + L\{t\,y'(t)\} - Y(s) = 0.$$

Note that

$$L\{t\,y'(t)\} = (-1)\frac{d}{ds}(sY(s) - y(0))$$

as $L\{t^n f(t)\} = (-1)^n \left(\frac{d}{ds}\right)^n L\{f(t)\}$.

Here $f(t) = y'(t)$.

Thus $s^2 Y(s) - sy(0) - y'(0) - \frac{d}{ds}(sY(s) - y(0)) - Y(s) = 0$

$\Rightarrow s^2 Y(s) - sy(0) - y'(0) - Y(s) - sY'(s) + 0 - Y(s) = 0$

$\Rightarrow s^2 Y(s) - sy(0) - y'(0) - 2Y(s) - sY'(s) = 0.$

Applying given initial conditions, we have

$s^2 Y(s) - 1 - 2Y(s) - sY'(s) = 0.$

$\Rightarrow -sY'(s) + (s^2 - 2)Y(s) - 1 = 0.$

$\Rightarrow Y'(s) - \frac{(s^2-2)}{s} Y(s) + \frac{1}{s} = 0.$

$$\Rightarrow \frac{d}{ds} Y(s) + \frac{2-s^2}{s} Y(s) = \frac{-1}{s} \quad \text{——— (1)}$$

Put $\frac{2-s^2}{s} = P$, $\frac{-1}{s} = Q$.

then

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which is 1st order linear nonhomogeneous DE

$\frac{dy}{dx} + P(x)y = Q(x)$

Integrating factor

$I.F = e^{\int P(x)dx}$

$$\frac{dy}{ds} + P(s)y = Q(s).$$

which is 1st order ordinary differential equation.

Its integrating factor is given by

$$I.F. = e^{\int P(s)ds} \quad \text{---------- (2)}$$

Note that $\int P(s)\,ds = \int \left(\frac{2-s^2}{s}\right) ds = \int \left(\frac{2}{s} - s\right) ds.$

$$= 2\int \frac{ds}{s} - \int s\,ds = 2\ln s - \frac{s^2}{2}$$

$$(2) \Rightarrow I.F. = e^{2\ln s - s^2/2} = e^{2\ln s} \cdot e^{-s^2/2} = e^{\ln s^2} e^{-s^2/2}$$

$$\Rightarrow I.F. = s^2 e^{-s^2/2}$$

Xing equation (1) by I.F, we get

$$s^2 e^{-s^2/2} \frac{dy}{ds} + \left(\frac{2-s^2}{s}\right) s^2 e^{-s^2/2} y(s) = \frac{-1}{s} \cdot s^2 e^{-s^2/2}$$

$$\Rightarrow s^2 e^{\frac{-s^2}{2}} \frac{dy}{ds} + 2s e^{\frac{-s^2}{2}} y(s) - s \cdot s^2 e^{\frac{-s^2}{2}} y(s) = -s e^{\frac{-s^2}{2}}$$

$$\Rightarrow \frac{d}{ds}\left\{s^2 e^{-s^2/2} y(s)\right\} = -s e^{-s^2/2}$$

$$\Rightarrow s^2 e^{\frac{-s^2}{2}} y(s) = -\int s e^{\frac{-s^2}{2}} ds + c = \int e^{\frac{-s^2}{2}} (-s)\,ds + c$$

$$= e^{\frac{-s^2}{2}} + c$$

$$\Rightarrow y(s) = \frac{1}{s^2} + \frac{c}{s^2} e^{s^2/2}$$

Taking lim $s \to \infty$, we get

$c = 0$

So $y(s) = 1/s^2$

Taking inverse laplace, we get

$y(t) = t.$

Anwan Shau

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The Gaussian Integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

and $\int_{0}^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

The Gamma Function.

The gamma function $\Gamma(x)$ is defined as.

$$\Gamma(x) = \int_{0}^{\infty} e^{-u} u^{x-1}\,du.$$

- $\Gamma(x+1) = x\,\Gamma(x)$.
- $\Gamma(1) = 1$
- $\Gamma(n) = (n-1)!$

Laplace transform of unit (Heaviside) step function.

By def. $L\{H(t-t_0)\} = \int_{0}^{\infty} e^{-st} H(t-t_0)\,dt$

$$= \int_{0}^{t_0} e^{-st} H(t-t_0)\,dt + \int_{t_0}^{\infty} e^{-st} H(t-t_0)\,dt$$

By definition of unit step function.

$$H(t-t_0) = \begin{cases} 1, & t \geq t_0 \\ 0, & t < t_0 \end{cases}$$

$$L\{H(t-t_0)\} = 0 + \int_{t_0}^{\infty} e^{-st}(1)\,dt = \int_{t_0}^{\infty} e^{-st}\,dt$$

$$= \frac{e^{-st}}{-s}\Big|_{t_0}^{\infty} = \frac{-1}{s}\{0 - e^{-st_0}\}.$$

Hence $L\{H(t-t_0)\} = \frac{e^{-st_0}}{s}$

if $t_0 = 0$ then $L\{H(t)\} = \frac{1}{s}$.

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Laplace transform of Logrithmic Function.

By defi: $L\{\ln t\} = \int_0^\infty e^{-st} \ln t \, dt$

Put $st = u \Rightarrow dt = \frac{du}{s}$

$t \to 0 \Rightarrow u \to 0$

$t \to \infty \Rightarrow u \to \infty$.

Then $L\{\ln t\} = \int_0^\infty e^{-u} \ln\left(\frac{u}{s}\right) \frac{du}{s}$

$= \frac{1}{s}\int_0^\infty e^{-u}(\ln u - \ln s)\, du$

$= \frac{1}{s}\int_0^\infty e^{-u} \ln u \, du - \frac{1}{s}\int_0^\infty e^{-u} \ln s \, du$.

$\Rightarrow L\{\ln t\} = \frac{1}{s}\int_0^\infty e^{-u} \ln u \, du - \frac{\ln s}{s}\int_0^\infty e^{-u} du$. ——— (1)

Note that $\int_0^\infty e^{-u} du = -e^{-u}\Big|_0^\infty = -(0-1) = 1$.

By definition of $\Gamma(x)$, we can write,

$$\Gamma(x+1) = \int_0^\infty e^{-u} u^x \, du.$$

Differentiating w.r.t. x, we get

$$\frac{d}{dx}\Gamma(x+1) = \int_0^\infty e^{-u} \frac{d}{dx} u^x \, du.$$

$$\Rightarrow \Gamma'(x+1) = \int_0^\infty e^{-u} u^x \ln u \, du.$$

Put $x = 0$, then

$$\Gamma'(1) = \int_0^\infty e^{-u} u^0 \ln u \, du$$

$$\Rightarrow \Gamma'(1) = \int_0^\infty e^{-u} \ln u \, du$$

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Thus equation ① becomes

$$L\{\ln t\} = \frac{1}{s}\Gamma'(1) - \frac{\ln s}{s} = \frac{1}{s}(\Gamma'(1) - \ln s).$$

where $\Gamma'(1)$ is constant called Euler's Constant and its approximately value is 0.577215665.

Laplace transform of the form $t^n f(t)$.

Theorem.

If $f(t)$ is a function of exponential order c, then

$$L\{t^n f(t)\} = (-1)^n \left(\frac{d}{ds}\right)^n F(s). \quad \text{for } s > a.$$

Proof.

By definition of Laplace transform

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

or $$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Diff. w.r.t s, we get.

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt.$$

$$= \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\,dt$$

اس لیے partial deri
کیونکہ اس میں s اور
t دو variable ہیں

$$= \int_0^\infty -t e^{-st} f(t)\,dt.$$

$$= -\int_0^\infty e^{-st}\, t f(t)\,dt.$$

$$\Rightarrow -\frac{d}{ds}F(s) = \int_0^\infty e^{-st}\, t f(t)\,dt.$$

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Again diff. w.r.t. s, we get,

$$-\frac{d^2}{ds^2}F(s) = \int_0^\infty -t e^{-st} t f(t)\,dt$$

$$\Rightarrow (-1)^2 \frac{d^2}{ds^2}F(s) = \int_0^\infty e^{-st} t^2 f(t)\,dt.$$

Similarly,

$$(-1)^3 \frac{d^3}{ds^3}F(s) = \int_0^\infty e^{-st} t^3 f(t)\,dt$$

Continuing in this way, we get,

$$(-1)^n \frac{d^n}{ds^n}F(s) = \int_0^\infty e^{-st} t^n f(t)\,dt.$$

or $$(-1)^n \frac{d^n}{ds^n}F(s) = L\{t^n f(t)\}.$$

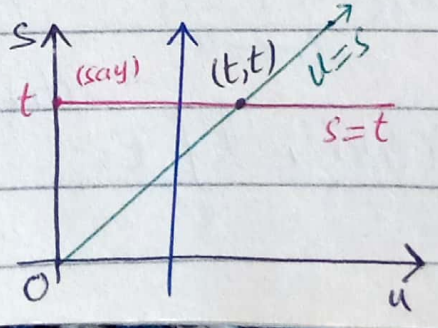
Prove that $(f * g) * h = f * (g * h)$ Associative law for convolution.

$$L.H.S = ((f*g)*h)(t) = \int_{s=0}^{t} (f*g)(s)\, h(t-s)\,ds$$

$$= \int_{s=0}^{t} \left(\int_{u=0}^{s} f(u) g(s-u)\,du \right) h(t-s)\,ds$$

$$= \int_{s=0}^{t} \int_{u=0}^{s} f(u) g(s-u) h(t-s)\,du\,ds.$$

$$\Rightarrow (f*g)*h = \int_{u=0}^{t} \int_{s=u}^{t} f(u) g(s-u) h(t-s)\,ds\,du \quad \text{(1)} \quad \because \text{change order of integ.}$$



For external limit we check the minimum and max value of u which are 0 and t respectively. For internal limit we apply vertical line test

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Put $s-u=\tau$ in ①,

$\Rightarrow ds = d\tau$ $\quad$ $\because$ u is constant for internal integral.

Then $(f*g)*h = \int_{u=0}^{t} f(u)\left\{\int_{s=u}^{t} g(s-u)\,h(t-s)\,ds\right\}du$.

$= \int_{u=0}^{t} f(u)\left\{\int_{\tau=0}^{t-u} g(\tau)\,h(t-u-\tau)\,d\tau\right\}du$.

$= \int_{u=0}^{t} f(u)\,(g*h)(t-u)\,du$

$= (f*(g*h))(t) = R.H.S.$

Hence $(f*g)*h = f*(g*h)$.

Second shifting (translation) Property.

It is stated as

2018/S $L^{-1}\{e^{-as}F(s)\} = H(t-a)f(t-a)$ $\because (a>0)$

It helps us to calculate the inverse Laplace transform of function of the form $e^{-as}F(s)$.

Proof.

By definition of Laplace transform

$L\{H(t-a)f(t-a)\} = \int_{t=0}^{\infty} e^{-st}H(t-a)f(t-a)\,dt$

Let $t-a=t_1 \Rightarrow dt=dt_1$

$t\to 0 \Rightarrow t_1 \to -a$

$t\to\infty \Rightarrow t_1 \to \infty$.

Then $L\{H(t-a)f(t-a)\} = \int_{t_1=-a}^{\infty} e^{-s(t_1+a)}H(t_1)f(t_1)\,dt_1$

$= \int_{-a}^{0} e^{-s(t_1+a)}H(t_1)f(t_1)\,dt_1 + \int_{0}^{\infty} e^{-s(t_1+a)}H(t_1)f(t_1)\,dt_1$

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By definition of unit (Heaviside) step function.

$$L\{H(t-a)f(t-a)\} = 0 + \int_0^\infty e^{-s(t_1+a)} f(t_1)\,dt_1$$

$$= e^{-as}\int_0^\infty e^{-st_1} f(t_1)\,dt_1$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)\,dt$$

{in definite integral variable can be changed}

$$= e^{-as} L\{f(t)\}.$$

$$= e^{-as} F(s).$$

$$\Rightarrow H(t-a)f(t-a) = L^{-1}\{e^{-as}F(s)\}.$$

Rutta

Example 1.

Find the Laplace transform of $J_0(t)$ where

$$J_0(t) = \frac{1}{\pi}\int_0^\pi \cos(t\sin\theta)\,d\theta.$$

Solution.

By definition of Laplace transform

$$L\{J_0(t)\} = \int_0^\infty e^{-st}\frac{1}{\pi}\left\{\int_0^\pi \cos(t\sin\theta)\,d\theta\right\}dt.$$

Change the order of integration, to get,

$$L\{J_0(t)\} = \frac{1}{\pi}\int_0^\pi\left\{\int_0^\infty e^{-st}\cos(t\sin\theta)\,dt\right\}d\theta.$$

$$= \frac{1}{\pi}\int_0^\pi I\,d\theta \quad \text{——— (1)}$$

where $I = \int_0^\infty e^{-st}\cos(t\sin\theta)\,dt$

Since $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

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Therefore,

$$I = \int_0^\infty e^{-st}\cos(t\sin\theta)\,dt = \int_0^\infty e^{(-s)t}\cos(\sin\theta)t\,dt$$

$$= \frac{e^{-st}}{s^2+\sin^2\theta}\left\{-s\cos(\sin\theta)t + \sin\theta\cdot\sin(\sin\theta)t\right\}\Big|_0^\infty$$

$$= \frac{e^{-st}}{s^2+\sin^2\theta}\left\{-s\cos(\sin\theta)t + \sin\theta\sin(\sin\theta)t\right\}\Big|_0^\infty$$

$$= 0 - \frac{1}{s^2+\sin^2\theta}\left\{-s(1) + 0\right\}$$

$$= \frac{-1}{s^2+\sin^2\theta}\{-s\} = \frac{s}{s^2+\sin^2\theta}$$

$$\Rightarrow I = \frac{s}{s^2+\sin^2\theta}$$

Thus equation ① becomes

$$L\{J_0(t)\} = \frac{1}{\pi}\int_0^\pi \frac{s}{s^2+\sin^2\theta}\,d\theta.$$

$$= \frac{s}{\pi}\int_{\theta=0}^{\pi} \frac{d\theta}{s^2+\sin^2\theta}$$

Put $\theta = \theta' + \frac{\pi}{2} \Rightarrow d\theta = d\theta'$

$\theta \to 0 \Rightarrow \theta' \to -\pi/2$

$\theta \to \pi \Rightarrow \theta' \to \pi/2$

Then $L\{J_0(t)\} = \frac{s}{\pi}\int_{\theta'=-\frac{\pi}{2}}^{\pi/2} \frac{d\theta'}{s^2+\sin^2(\frac{\pi}{2}+\theta')}$

$$= \frac{2s}{\pi}\int_0^{\pi/2} \frac{d\theta'}{s^2+\cos^2\theta'}$$

نوٹ:-
یہ سوال sine کو cos میں change کیے بغیر بھی ہو سکتا ہے

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$$= \frac{2s}{\pi}\int_0^{\pi/2} \frac{d\theta}{s^2+\cos^2\theta}$$ ∵ variable can changed in definite integral.

$$= \frac{2s}{\pi}\int_0^{\pi/2} \frac{d\theta}{\cos^2\theta(1+s^2\sec^2\theta)} = \frac{2s}{\pi}\int_0^{\pi/2} \frac{\sec^2\theta d\theta}{1+s^2\sec^2\theta}$$

$$= \frac{2s}{\pi}\int_0^{\pi/2} \frac{\sec^2\theta d\theta}{1+s^2(1+\tan^2\theta)}$$

$$= \frac{2s}{\pi}\int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{1+s^2(1+\tan^2\theta)}$$

Put $u = \tan\theta \Rightarrow du = \sec^2\theta d\theta$

$\theta \to 0 \Rightarrow u \to 0$

$\theta \to \frac{\pi}{2} \Rightarrow u \to \infty$.

Then $L\{J_0(t)\} = \frac{2s}{\pi}\int_{u=0}^{\infty} \frac{du}{1+s^2(1+u^2)}$

$$= \frac{2s}{\pi s^2}\int_0^{\infty} \frac{du}{\frac{1}{s^2}+1+u^2}$$

$$= \frac{2}{\pi s}\int_0^{\infty} \frac{du}{a^2+u^2}$$

where $\frac{1}{s^2}+1 = a^2 \Rightarrow \frac{1+s^2}{s^2} = a^2$

$$= \frac{2}{\pi s}\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)\Big|_0^{\infty}$$

$$= \frac{2}{a\pi s}\left\{\frac{\pi}{2}-0\right\} = \frac{1}{as} = \frac{s}{s\sqrt{1+s^2}}$$

$$\Rightarrow L\{J_0(t)\} = \frac{1}{\sqrt{1+s^2}}$$

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Example 2.

Given that Bessel functions of the 1st kind and +ve integral order satisfy the recurrence relations

$$J_1 = -J_0', \quad J_{n+1} = J_{n-1} - 2J_n', \quad n \geq 1$$

with $J_0(0) = 1$, $J_n(0) = 0$, $n > 0$,

Show that

$$L\{J_n(t)\} = \frac{(\sqrt{s^2+1} - s)^n}{\sqrt{s^2+1}}$$

Also find the Laplace transform for $J_0(at)$, $a > 0$.

Solution.

By definition of Laplace transform

$$L\{J_1(t)\} = -L\{J_0'(t)\}.$$

$$\Rightarrow F_1(s) = -\{+sF_0(s) - J_0(0)\}.$$

$$\Rightarrow F_1(s) = -sF_0(s) + 1 \quad \because J_0(0) = 1.$$

$$\Rightarrow F_1(s) = -\frac{s}{\sqrt{1+s^2}} + 1 \quad \because \text{by exampl.}$$

$$L\{J_0(t)\} = F_0(s)$$

$$F_0(s) = \frac{1}{\sqrt{1+s^2}}$$

$$\Rightarrow F_1(s) = \frac{\sqrt{1+s^2} - s}{\sqrt{1+s^2}}$$

or $$F_1(s) = \frac{\sqrt{s^2+1} - s}{\sqrt{s^2+1}}.$$

Now by 2nd recurrence relation

$$J_2 = J_0 - 2J_1'$$

$$\Rightarrow L\{J_2(t)\} = L\{J_0\} - 2L\{J_1'(t)\}.$$

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$$\Rightarrow F_2(s) = F_0(s) - 2\left(sF_1(s) - J_1(0)\right)$$

$$= \frac{1}{\sqrt{s^2+1}} - 2s\left(\frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}}\right) \quad \because J_1(0)=0$$

$$= \frac{1 - 2s\sqrt{s^2+1} + 2s^2}{\sqrt{s^2+1}}$$

$$= \frac{(s^2+1) + s^2 - 2s\sqrt{s^2+1}}{\sqrt{s^2+1}}$$

$$F_2(s) = \frac{\left(\sqrt{s^2+1}-s\right)^2}{\sqrt{s^2+1}}$$

Continuing in this way, we get

$$F_n(s) = \frac{\left(\sqrt{s^2+1}-s\right)^n}{\sqrt{s^2+1}}$$

$$\text{or } L\{J_n(t)\} = \frac{\left(\sqrt{s^2+1}-s\right)^n}{\sqrt{s^2+1}}.$$

Since we know that $L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$, $a>0$

Therefore,

$$L\{J_n(at)\} = \frac{\left(\sqrt{s^2+a^2}-s\right)^n}{a^n\sqrt{s^2+a^2}}$$

$$\Rightarrow L\{J_0(at)\} = \frac{\left(\sqrt{s^2+a^2}-s\right)^0}{a^0\sqrt{s^2+a^2}} = \frac{1}{\sqrt{s^2+a^2}}$$

Example 3.

Find $L\left\{\frac{e^{at}-\cos bt}{t}\right\}$ and deduce that

$$L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{2}\ln\left(\frac{\sqrt{s^2+4}}{s}\right), \quad s>0.$$

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Solution.

We know that

$$L\{e^{at}-\cos bt\} = L\{e^{at}\} - L\{\cos bt\}.$$

$$= \frac{1}{s-a} - \frac{s}{s^2+b^2}.$$

Since

$$L\left\{\frac{f(t)}{t}\right\} = \int_{s_1=s}^{\infty} F(s_1)\,ds_1$$

Therefore,

$$L\left\{\frac{e^{at}-\cos bt}{t}\right\} = \int_{s_1=s}^{\infty}\left[\frac{1}{s_1-a} - \frac{s}{s_1^2+b^2}\right] ds_1.$$

$$= \left[\ln|s_1-a| - \frac{1}{2}\ln|s_1^2+b^2|\right]_{s_1=s}^{\infty}$$

$$= 0 - \left\{\ln|s-a| - \frac{1}{2}\ln|s^2+b^2|\right\}.$$

$$= -\ln|s-a| + \ln\sqrt{s^2+b^2}.$$

$$\Rightarrow L\left\{\frac{e^{at}-\cos bt}{t}\right\} = \ln\frac{\sqrt{s^2+b^2}}{|s-a|}$$

put $a=0$ and $b=2$, then

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \ln\frac{\sqrt{s^2+4}}{s}$$

$$\Rightarrow L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{2}\ln\frac{\sqrt{s^2+4}}{s}.$$

Example 4.

Show that $L\left\{\frac{f(t)}{t+a}\right\} = e^{as}\int_{s_1=s}^{\infty} e^{-s_1 a} F(s_1)\,ds_1.$

Solution.

Let $I = L\left\{\frac{f(t)}{t+a}\right\}$ and $D = \frac{d}{ds}$

Then

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$$I = \int_{t=0}^{\infty} e^{-st} \frac{f(t)}{t+a}\, dt.$$

$$\Rightarrow (D-a)I = (D-a)\int_0^{\infty} e^{-st} \frac{f(t)}{t+a}\, dt.$$

$$= D\int_0^{\infty} e^{-st} \frac{f(t)}{t+a}\, dt - a\int_0^{\infty} e^{-st} \frac{f(t)}{t+a}\, dt.$$

$$= \frac{d}{ds}\int_0^{\infty} e^{-st} \frac{f(t)}{t+a}\, dt - a\int_0^{\infty} e^{-st} \frac{f(t)}{t+a}\, dt.$$

$$= \int_0^{\infty} \frac{\partial}{\partial s} e^{-st} \frac{f(t)}{t+a}\, dt - a\int_0^{\infty} e^{-st} \frac{f(t)}{t+a}\, dt.$$

$$= -\int_0^{\infty} e^{-st} \frac{t\, f(t)}{t+a}\, dt - a\int_0^{\infty} e^{-st} \frac{f(t)}{t+a}\, dt.$$

$$= -\int_0^{\infty} e^{-st} \frac{(t+a)\, f(t)}{t+a}\, dt$$

$$= -\int_0^{\infty} e^{-st} f(t)\, dt = -F(s).$$

$$\Rightarrow \frac{d}{ds}I - aI = -F(s). \quad \text{——— (1)}$$

which 1st order linear ODE.

The integrating factor is given by

$$I.F. = \exp\left(\int(-a)\, ds\right) = e^{-\int a\, ds} = e^{-as}$$

Xing (1) by I.F we get.

$$e^{-as}\frac{dI}{ds} - e^{-as} a I = -e^{-as} F(s).$$

$$\Rightarrow \frac{d}{ds}\left(e^{-as} I\right) = -e^{as} F(s).$$

Integrating, we get.

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$$e^{-as}\bar{I} = -\int e^{-as} F(s)\,ds + G_1 \qquad \text{——(2)}$$

To calculate the constant G_1, we take the limit as $s \to \infty$

$$0 = -\int_{s_0}^{\infty} e^{-as} F(s)\,ds + G_1$$

$$\Rightarrow G_1 = \int_{s_0}^{\infty} e^{-as} F(s)\,ds$$

Thus equation (2) becomes

$$e^{-as}\bar{I} = \int_{s}^{\infty} e^{-as_1} F(s_1)\,ds_1$$

$$\Rightarrow \bar{I} = e^{as}\int_{s}^{\infty} e^{-as_1} F(s_1)\,ds_1$$

Hence, $$\mathcal{L}\left\{\frac{f(t)}{t+a}\right\} = e^{as}\int_{s}^{\infty} e^{-as_1} F(s_1)\,ds_1.$$

Complementry Error Function.

$$erfc(t) = \frac{2}{\sqrt{\pi}}\int_{t}^{\infty} e^{-x^2}\,dx.$$

Error Function.

$$erf(t) = \frac{2}{\sqrt{\pi}}\int_{0}^{t} e^{x^2}\,dx.$$

The complementry error function can be written as

$$erfc(t) = \frac{2}{\sqrt{\pi}}\int_{t}^{\infty} e^{-x^2}\,dx.$$

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$$= \frac{2}{\sqrt{\pi}}\left[\int_t^0 e^{-x^2}dx + \int_0^\infty e^{-x^2}dx\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[-\int_0^t e^{-x^2}dx + \int_0^\infty e^{-x^2}dx\right].$$

$$= \frac{2}{\sqrt{\pi}}\left[-\int_0^t e^{-x^2}dt + \frac{\sqrt{\pi}}{2}\right] \quad \because \text{using Gaussian integral.}$$

$$= -\frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dt + 1$$

$$= -erf(t) + 1 \quad = 1 - erf(t).$$

Example.

Prove that $L\{erfc(t^{1/2})\} = \frac{1}{s} - \frac{1}{s\sqrt{s+1}}$

where $erfc(t^{1/2}) = 1 - erf(t^{1/2})$.

Solution.

As $erfc(t^{1/2}) = 1 - erf(t^{1/2})$.

so $L\{erfc(\sqrt{t})\} = L\{1\} - L\{erf(\sqrt{t})\}.$

$$= \frac{1}{s} - L\left\{\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}dx\right\}. \quad \text{——(1)}$$

Note that

$$erf(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}dx.$$

Put $x^2 = u \Rightarrow 2xdx = du$

or $dx = \frac{du}{2x} = \frac{du}{2\sqrt{u}}$

$x \to 0 \Rightarrow u \to 0$

$x \to \sqrt{t} \Rightarrow u \to t$

Then $erf(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^t \frac{e^{-u}}{2\sqrt{u}}du$

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$$= \frac{1}{\sqrt{\pi}} \int_0^t \frac{e^{-u}\,du}{\sqrt{u}}$$

$$\Rightarrow L\{erf(\sqrt{t})\} = \frac{1}{\sqrt{\pi}} L\left\{\int_0^t \frac{e^{-u}\,du}{\sqrt{u}}\right\}$$

Since we know that

$$L\left\{\int_0^t f(\bar{t})\,d\bar{t}\right\} = \frac{1}{s} L\{f(t)\}.$$

Therefore,

$$L\{erf(\sqrt{t})\} = \frac{1}{s\sqrt{\pi}} L\left\{\frac{e^{-t}}{\sqrt{t}}\right\} = \frac{1}{s\sqrt{\pi}} L\{e^{-t} t^{-1/2}\}$$

$$= \frac{1}{s\sqrt{\pi}} L\{t^{-1/2}\}_{s \to s+1}$$ by 1st shifting prop.

$$= \frac{1}{s\sqrt{\pi}} \sqrt{\frac{\pi}{s+1}}$$ see page 13

$$\Rightarrow L\{erf(\sqrt{t})\} = \frac{1}{s\sqrt{s+1}}$$

Now equation (1) becomes

$$L\{erfc(t^{1/2})\} = \frac{1}{s} - \frac{1}{s\sqrt{s+1}}$$ as required

Also note that

$$L\{e^t erfc\sqrt{t}\} = \left\{\frac{1}{s} - \frac{1}{s\sqrt{s+1}}\right\}_{s \to s-1} = L\{erfc\sqrt{t}\}_{s \to s-1}$$ by 1st shifting prop.

$$= \frac{1}{s-1} - \frac{1}{(s-1)\sqrt{s}}$$

$$= \frac{\sqrt{s} - 1}{\sqrt{s}(s-1)} = \frac{(\sqrt{s}-1)(\sqrt{s}+1)}{\sqrt{s}(s-1)(\sqrt{s}+1)}$$

$$= \frac{s-1}{\sqrt{s}(s-1)\sqrt{s}+\sqrt{s}+1} = \frac{1}{s+\sqrt{s}}$$

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Example.

Use Laplace transform to solve the problem

$u'' - au = f(t)$, $u(0) = u_0$, $u'(0) = u_1$

Solution.

Given that $u'' - au = f(t)$

Taking Laplace transform on both sides, we get,

$$L\{u''\} - aL\{u\} = L\{f(t)\}$$

$$\Rightarrow s^2U(s) - su(0) - u'(0) - aU(s) = F(s)$$

$$\Rightarrow (s^2 - a)U(s) = F(s) + su_0 + u_1$$

$$\Rightarrow U(s) = \frac{F(s)}{s^2 - a} + \frac{su_0}{s^2 - a} + \frac{u_1}{s^2 - a}$$

$$= \frac{1}{\sqrt{a}} F(s) \cdot \frac{\sqrt{a}}{s^2 - (\sqrt{a})^2} + u_0 \frac{s}{s^2 - (\sqrt{a})^2} + \frac{u_1}{\sqrt{a}} \frac{\sqrt{a}}{s^2 - (\sqrt{a})^2}$$

Taking L^{-1} on both sides, we get,

$$L^{-1}\{U(s)\} = \frac{1}{\sqrt{a}} L^{-1}\left\{F(s) \cdot \frac{\sqrt{a}}{s^2 - (\sqrt{a})^2}\right\} + u_0 L^{-1}\left\{\frac{s}{s^2 - (\sqrt{a})^2}\right\}$$

$$+ \frac{u_1}{\sqrt{a}} L^{-1}\left\{\frac{\sqrt{a}}{s^2 - (\sqrt{a})^2}\right\}$$

$$\Rightarrow u(t) = \frac{1}{\sqrt{a}} f(t) * \sinh\sqrt{a}t + u_0 \cosh\sqrt{a}t + \frac{u_1}{\sqrt{a}} \sinh\sqrt{a}t$$

Now we can find the inverse Laplace transform as follows

$$L^{-1}\{F(s)\} = f(t) = \sum_{i=1}^{n} R_i$$

where R_i = residue of $e^{st}F(s)$ at the pole $s = s_i$.

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Example 1.

Use the Laplace inversion (Fouries Mellin) integral to evaluate $L^{-1}\left\{\frac{s^3+2s^2+1}{s^2(s^2+1)}\right\}$.

Solution.

Given that $L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s^3+2s^2+1}{s^2(s^2+1)}\right\}$

$\Rightarrow F(s) = \frac{s^3+2s^2+1}{s^2(s^2+1)}$.

$\Rightarrow e^{st}F(s) = e^{st}\frac{s^3+2s^2+1}{s^2(s^2+1)}$.

The function $e^{st}F(s)$ has a pole of order 2 at $s=0$ and it has simple poles at $s=\pm i$.

Therefore by Laplace inversion integral theorem

$$f(t) = \sum_{i=1}^{3} R_i = R_1 + R_2 + R_3 \quad \text{——— (1)}$$

where R_1 = residue of function $e^{st}F(s)$ at $s=0$

$$= \lim_{s\to 0}\frac{d}{ds}\left\{(s-0)^2 e^{st}F(s)\right\}.$$

$$= \lim_{s\to 0}\frac{d}{ds}\left\{s^2 e^{st}\frac{s^3+2s^2+1}{s^2(s^2+1)}\right\}$$

$$= \lim_{s\to 0}\frac{d}{ds}\left\{e^{st}(s^3+2s^2+1)\cdot\frac{1}{s^2+1}\right\}.$$

$$= \lim_{s\to 0}\left\{e^{st}(s^3+2s^2+1)\frac{(-2s)}{(s^2+1)^2} + \right.$$

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$$e^{st}(3s^2+4s)\frac{1}{s^2+1} + te^{st}(s^3+2s^2+1)\frac{1}{s^2+1}\Big\}.$$

$$\Rightarrow R_1 = \lim_{s\to 0}\Big\{-\frac{2se^{st}(s^3+2s^2+1)}{s^2+1} + \frac{e^{st}(3s^2+4s)}{s^2+1} + \frac{te^{st}(s^3+2s^2+1)}{s^2+1}\Big\}.$$

$$= 0 + 0 + \frac{te^{0}(0+0+1)}{0+1} = t$$

$$\Rightarrow \boxed{R_1 = t}$$

R_2 = residue of $e^{st}F(s)$ at $s=+i$

$$= \lim_{s\to i}\{(s-i)e^{st}F(s)\} = \lim_{s\to i}\Big\{\frac{(s-i)e^{st}(s^3+2s^2+1)}{s^2(s+i)(s-i)}\Big\}$$

$$= \lim_{s\to i}\Big\{\frac{e^{st}(s^3+2s^2+1)}{s^2(s+i)}\Big\}.$$

$$= \frac{e^{it}(-i+2(-1)+1)}{-1(i+i)} = \frac{e^{it}(-i-1)}{-2i}$$

$$= \frac{1}{2i}e^{it}(1+i) = \frac{1}{2i}e^{it}(1+i)\times\frac{-i}{-i}$$

$$= \frac{1}{2(1)}e^{it}(-i+1).$$

$$\Rightarrow \boxed{R_2 = \frac{1}{2}e^{it}(1-i)}$$

and R_3 = residue of $e^{st}F(s)$ at $s=-i$

$$= \lim_{s\to -i}\Big\{\frac{(s+i)e^{st}(s^3+2s^2+1)}{s^2(s+i)(s-i)}\Big\}.$$

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$$= \lim_{s \to -i} \left\{ \frac{e^{st}(s^3+2s^2+1)}{s^2(s-i)} \right\}.$$

$$= \frac{e^{-it}(-i-2+1)}{-1(-2i)} = \frac{1}{2i} e^{-it}(-1-i).$$

$$= \frac{1}{2i} e^{-it}(-1-i) \times \frac{-i}{-i} = \frac{1}{2} e^{-it}(i+1)$$

$$\Rightarrow \boxed{R_3 = \frac{1}{2} e^{-it}(1+i)}$$

Now equation ① becomes

$$f(t) = t + \frac{1}{2} e^{it}(1-i) + \frac{1}{2} e^{-it}(1+i).$$

$$= t + \frac{1}{2}\{e^{it} - ie^{it} + e^{-it} + ie^{-it}\}.$$

$$= t + \frac{1}{2}\{\cos t + i\sin t - i(\cos t + i\sin t) + \cos t - i\sin t + i(\cos t - i\sin t)\}.$$

$$= t + \frac{1}{2}\{2\cos t - i\cos t + \sin t + i\cos t + \sin t\}$$

$$= t + \frac{1}{2}\{2\cos t + 2\sin t\}.$$

$$f(t) = t + \cos t + \sin t.$$

Hence

$$L^{-1}\left\{ \frac{s^3+2s^2+1}{s^2(s^2+1)} \right\} = t + \cos t + \sin t.$$

Example 2.

Use the inverson integral method to calculate the Laplace transform of

(i) $F(s) = \dfrac{2s+1}{s(s^2+1)}$.

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(ii) $F(s) = \dfrac{1}{s^2(s+1)}$.

Solution.

(i)

Given that $F(s) = \dfrac{2s+1}{s(s^2+1)}$.

$\Rightarrow e^{st} F(s) = e^{st} \dfrac{2s+1}{s(s^2+1)}$.

The function $e^{st}F(s)$ has poles of order one or simple poles at $s=0$ and $s=\pm i$.

Therefore, by Laplace inversion integral.

$f(t) = R_1 + R_2 + R_3$ ——— (1)

where R_1 = residue of $e^{st}F(s)$ at $s=0$.

$= \lim_{s\to 0} \left\{ s\, e^{st} \dfrac{2s+1}{s(s^2+1)} \right\} = \lim_{s\to 0} \left\{ e^{st} \dfrac{(2s+1)}{s^2+1} \right\}$.

$\Rightarrow \boxed{R_1 = 1}$

R_2 = residue of $e^{st}F(s)$ at $s=i$

$= \lim_{s\to i} \left\{ (s-i)\, e^{st} \dfrac{(2s+1)}{s(s-i)(s+i)} \right\}$.

$= \lim_{s\to i} \left\{ e^{st} \dfrac{2s+1}{s(s+i)} \right\} = \dfrac{e^{it}(2i+1)}{i(2i)}$

$\Rightarrow R_2 = \dfrac{-1}{2} e^{it}(1+2i)$.

and R_3 = residue of $e^{st}F(s)$ at $s=-i$.

$= \lim_{s\to -i} \left\{ (s+i)\, e^{st} \dfrac{(2s+1)}{s(s-i)(s+i)} \right\}$

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$$= \lim_{s\to -i}\left\{ \frac{e^{st}(2s+1)}{s(s-i)} \right\} = \frac{e^{-it}(-2i+1)}{-i(-2i)}$$

$$\Rightarrow R_3 = -\frac{1}{2} e^{-it}(1-2i).$$

Now equation (1) becomes

$$f(t) = 1 - \frac{1}{2} e^{it}(1+2i) - \frac{1}{2} e^{-it}(1-2i)$$

$$= 1 - \frac{1}{2}\{(\cos t + i\sin t)(1+2i) + (\cos t - i\sin t)(1-2i)\}$$

$$= 1 - \frac{1}{2}\{\cos t + 2i\cos t + i\sin t - 2\sin t + \cos t - 2i\cos t - i\sin t - 2\sin t\}$$

$$= 1 - \frac{1}{2}\{2\cos t - 4\sin t\}.$$

$$= 1 - \cos t + 2\sin t.$$

(ii)

Given that $F(s) = \frac{1}{s^2(s+1)}$

$$\Rightarrow e^{st} F(s) = \frac{e^{st}}{s^2(s+1)}$$

The function $e^{st}F(s)$ has pole of order 2 at $s=0$ and pole of order one at $s=-1$.

Therefore by Laplace inversion integral.

$$f(t) = R_1 + R_2 \quad \text{---------- (1)}$$

where R_1 = residue of $e^{st}F(s)$ at $s=0$.

$$= \lim_{s\to 0} \frac{d}{ds}\left\{ s^2 \frac{e^{st}}{s^2(s+1)} \right\}$$

$$= \lim_{s\to 0} \frac{d}{ds}\left\{ \frac{e^{st}}{s+1} \right\}.$$

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$$= \lim_{s\to 0}\left\{\frac{(s+1)te^{st} - e^{st}(1)}{(s+1)^2}\right\} = \lim_{s\to 0}\left\{\frac{ste^{st} + te^{st} - e^{st}}{(s+1)^2}\right\}$$

$$= \frac{0+t-1}{1} = t-1.$$

and R_2 = residue of $e^{st}F(s)$ at $s=-1$.

$$= \lim_{s\to -1}\left\{\cancel{(s+1)}\frac{e^{st}}{s^2\cancel{(s+1)}}\right\} = \lim_{s\to -1}\left\{\frac{e^{st}}{s^2}\right\}.$$

$$\Rightarrow R_2 = \frac{e^{-t}}{1} = e^{-t}.$$

Now equation ① becomes

$$f(t) = t-1+e^{-t}.$$

Application to PDEs.

$$L\left\{\frac{\partial u(x,t)}{\partial x}\right\} = \frac{\partial}{\partial x}U(x,s)$$

{Laplace کیونکہ t کے according ہے}

and $L\left\{\frac{\partial u(x,t)}{\partial t}\right\} = sU(x,s) - u(x,0).$

Example 1. 2012, 2018/S $(u(x,0) = 1-\sin\pi x)$

Use Laplace transform method to solve

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0<x<1, \quad 0\le t<\infty.$$

$$u(0,t)=1, \quad u(1,t)=1, \quad t>0$$

$$u(x,0) = 1+\sin\pi x, \quad 0<x<1.$$

Solution.

Given that $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0<x<1$ —— ①

$$u(0,t)=1, \quad u(1,t)=1 \quad \text{—— ②}$$

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and $u(x,0) = 1 + 5\sin\pi x$, $0<x<1$ ——— (3).

Taking Laplace transform of (1) and (2), we get,

(1) $\Rightarrow L\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} = L\left\{\frac{\partial u(x,t)}{\partial t}\right\}$.

$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) = sU(x,s) - u(x,0)$.

$\Rightarrow \frac{d^2}{dx^2} U(x,s) = sU(x,s) - 1 - 5\sin\pi x$ $\because$ using (3)

$\Rightarrow D^2 U(x,s) - sU(x,s) = -1 - 5\sin\pi x$ $\because D^2 = \frac{d^2}{dx^2}$

——— (4)

(2) $\Rightarrow L\{u(0,t)\} = L\{1\}$; $L\{u(1,t)\} = L\{1\}$.

$\Rightarrow U(0,s) = \frac{1}{s}$; $U(1,s) = \frac{1}{s}$ ——— (5)

For solving equation the characteristic equation is

$D^2 - s = 0 \Rightarrow D^2 = s$

$\Rightarrow D = \pm\sqrt{s}$

So $U_c(x,s) = C_1 e^{\sqrt{s}x} + C_2 e^{-\sqrt{s}x}$ ——— (6)

Now we find $U_p(x,s)$ as follows.

$$U_p(x,s) = \frac{-1-5\sin\pi x}{D^2-s} = \frac{-1}{D^2-s} - \frac{5\sin\pi x}{D^2-s}$$

$$= \frac{-1}{-s(1-\frac{D^2}{s})} - \frac{5\sin\pi x}{-\pi^2-s} \quad \because D^2 = -(\text{coefficient of } x)^2$$

$$= \frac{1}{s}\left(1-\frac{D^2}{s}\right)^{-1}(1) + \frac{5\sin\pi x}{\pi^2+s}$$

$$= \frac{1}{s}\left(1+\frac{D^2}{s}+\text{terms having higher powers of } D\right)(1).$$

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$$+\frac{\sin \pi x}{\pi^2+s}$$

$$=\frac{1}{s}(1)+\frac{\sin \pi x}{\pi^2+s} = \frac{1}{s}+\frac{\sin \pi x}{\pi^2+s} \quad \text{——} (7)$$

Thus the general solution is

$$U(x,s) = U_c(x,s) + U_p(x,s)$$

$$\Rightarrow U(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{1}{s} + \frac{\sin \pi x}{\pi^2+s} \quad \because \text{by } (6) \text{ and } (7)$$

Applying conditions which are given in (5), we get

$$U(0,s)=\frac{1}{s} \Rightarrow c_1 + c_2 + \frac{1}{s} = \frac{1}{s}$$

$$\Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2 \quad \text{——} (8)$$

$$U(1,s)=\frac{1}{s} \Rightarrow c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} + \frac{1}{s} = \frac{1}{s}$$

$$\Rightarrow c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} = 0$$

$$(9) \Rightarrow -c_2 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} = 0 \quad \because \text{by } (8)$$

$$\Rightarrow c_2 (e^{-\sqrt{s}} - e^{+\sqrt{s}}) = 0$$

$$\Rightarrow c_2 = 0 \quad \because e^{-\sqrt{s}} + e^{\sqrt{s}} \neq 0.$$

$$(8) \Rightarrow c_1 = 0$$

Thus $U(x,s) = \frac{1}{s} + \frac{\sin \pi x}{\pi^2+s}$

Taking L^{-1} on both sides, we get

$$L^{-1}\{U(x,s)\} = L^{-1}\{\frac{1}{s}\} + L^{-1}\{\frac{\sin \pi x}{\pi^2+s}\}$$

$$\Rightarrow u(x,t) = 1 + \sin \pi x \, L^{-1}\{\frac{1}{s+\pi^2}\}$$

$$= 1 + \sin \pi x \, e^{-\pi^2 t}$$

$$= 1 + e^{-\pi^2 t} \sin \pi x.$$

required solution.

$\sin \pi x$ ہے constant
ادھر کیونکہ اس
میں s نہیں ہے

Example 2.

Solve the problem by Laplace transform method

$$u_{tt}(x,t) = a^2 u_{xx}(x,t), \quad t>0, x>0 \quad \text{——} (1)$$

$$u(x,0) = u_t(x,0) = 0 \quad \text{——} (2)$$

$$u(0,t) = f(t); \lim_{x\to\infty} u(x,t) = 0 \quad \text{——} (3)$$

Solution.

Taking Laplace transform of equation (1), we get,

$$L\left\{\frac{\partial^2 u(x,t)}{\partial t^2}\right\} = a^2 L\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\}.$$

$$\Rightarrow s^2 U(x,s) - s u(x,0) - u_t(x,0) = a^2 \left\{\frac{\partial^2}{\partial x^2} U(x,s)\right\}$$

$$\Rightarrow s^2 U(x,s) - 0 - 0 = a^2 \frac{\partial^2 U(x,s)}{\partial x^2} \quad \because \text{using } (2)$$

$$\Rightarrow \frac{s^2}{a^2} U(x,s) = \frac{\partial^2 U(x,s)}{\partial x^2}$$

$$\Rightarrow \frac{d^2}{dx^2} U(x,s) - \frac{s^2}{a^2} U(x,s) = 0.$$

The characteristic equation is

$$D^2 - \frac{s^2}{a^2} = 0 \quad \text{where } D^2 = \frac{d^2}{dx^2}.$$

$$\Rightarrow D^2 = \frac{s^2}{a^2} \Rightarrow D = \pm \frac{s}{a}.$$

$$\Rightarrow U(x,s) = C_1 e^{\frac{s}{a}x} + C_2 e^{-\frac{s}{a}x} \quad \text{——} (4)$$

Taking Laplace transform on both sides of (3), we get,

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$$L\{u(0,t)\} = L\{f(t)\} \; ; \; \lim_{x\to\infty} L\{u(x,t)\} = 0.$$

$$\Rightarrow U(0,s) = F(s) \; ; \; \lim_{x\to\infty} U(x,s) = 0 \quad \text{——} (5)$$

$(4) \Rightarrow U(0,s) = C_1 + C_2$

$(5) \Rightarrow C_1 + C_2 = F(s) \quad \text{——} (6)$

and $\lim_{x\to\infty} U(x,s) = C_1 \lim_{x\to\infty} e^{\frac{s}{a}x} + C_2 \lim_{x\to\infty} e^{-\frac{s}{a}x}$

$$\Rightarrow 0 = C_1 \lim_{x\to\infty} e^{\frac{s}{a}x} + 0.$$

$\Rightarrow C_1 = 0$ because $\lim_{x\to\infty} e^{\frac{s}{a}x} \neq 0.$

$(6) \Rightarrow C_2 = F(s).$

Thus

$(4) \Rightarrow U(x,s) = F(s)\, e^{-\frac{s}{a}x}.$

Taking L^{-1} on both sides, we get,

$$L^{-1}\{U(x,s)\} = L^{-1}\{F(s)\, e^{-\frac{sx}{a}}\}$$

$\Rightarrow u(x,t) = H(t-\frac{x}{a}) f(t-x/a)$ by second shifting prop.

where

$$H(t-\tfrac{x}{a}) f(t-\tfrac{x}{a}) = \begin{cases} f(t-x/a), & t > x/a \\ 0, & t < x/a. \end{cases}$$

Example 3. 2016,

Solve the problem using the Laplace transform formalism.

$$u_{tt}(x,t) = a^2 u_{xx}(x,t) - 9 \quad \text{——} (1)$$

$$u(x,0) = u_t(x,0) = 0 \quad \text{——} (2)$$

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$u(0,t) = 0$, $\lim_{x\to\infty} u_x(x,t) = 0$ ——— (3)

Solution.

Taking Laplace transform on both sides of (1), we get,

$$L\left\{\frac{\partial^2}{\partial t^2}u(x,t)\right\} = a^2 L\left\{\frac{\partial^2}{\partial x^2}u(x,t)\right\} - g\,L\{1\}$$

$$\Rightarrow s^2U(x,s) - su(x,0) - u_t(x,0) = a^2\frac{\partial^2}{\partial x^2}U(x,s) - g\cdot\frac{1}{s}$$

$$\Rightarrow s^2U(x,s) = a^2\frac{d^2}{dx^2}U(x,s) - g/s \quad \because \text{using (2)}$$

$$\Rightarrow a^2\frac{d^2}{dx^2}U(x,s) - s^2U(x,s) = \frac{g}{s}$$

$$\Rightarrow D^2U(x,0) - \frac{s^2}{a^2}U(x,s) = \frac{g}{a^2 s}$$

The characteristic equation is

$$D^2 - \frac{s^2}{a^2} = 0$$

$$\Rightarrow D = \pm s/a$$

$$\Rightarrow U_c(x,s) = C_1 e^{\frac{s}{a}x} + C_2 e^{-\frac{s}{a}x} \quad \text{——— (4)}.$$

Now we find particular integral as follows.

$$U_p(x,s) = \frac{g/a^2 s}{D^2 - \frac{s^2}{a^2}} = \frac{g}{a^2 s}\cdot\frac{1}{D^2 - \frac{s^2}{a^2}}e^{0x}.$$

$$= \frac{g}{a^2 s}\cdot\frac{1}{0 - \frac{s^2}{a^2}} \quad \because D^2 = -(\text{Coeffi. of } x)^2$$

$$= \frac{-g}{a^2 s}\cdot\frac{a^2}{s^2}$$

$$\Rightarrow U_p(x,s) = \frac{-g}{s^3} \quad \text{——— (5)}$$

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The general solution is

$$U(x,s) = U_c(x,s) + U_p(x,s)$$

$$\Rightarrow U(x,s) = C_1 e^{\frac{s}{a}x} + C_2 e^{-s/a\, x} - \frac{g}{s^3} \quad \text{—— (6) by (4) \& (5)}$$

Taking Laplace of (3), we have,

$$U(0,s) = 0 \quad , \quad \lim_{x\to\infty} U_x(x,s) = 0$$

(derivative [illegible] x)

$$(6) \Rightarrow U(0,s) = C_1 + C_2 - g/s^3 = 0$$

$$\Rightarrow C_1 + C_2 = \frac{g}{s^3} \quad \text{—— (7)}$$

Also

$$(6) \Rightarrow U_x(x,s) = C_1 \frac{s}{a} e^{\frac{s}{a}x} - C_2 \frac{s}{a} e^{-\frac{s}{a}x}$$

$$\Rightarrow \lim_{x\to\infty} U_x(x,s) = C_1 \lim_{x\to\infty} \frac{s}{a} e^{\frac{s}{a}x} - C_2 \lim_{x\to\infty} \frac{s}{a} e^{-\frac{s}{a}x}$$

$$\Rightarrow 0 = C_1 \lim_{x\to\infty} \frac{s}{a} e^{\frac{s}{a}x}$$

$$\Rightarrow C_1 = 0 \quad \because \frac{s}{a} \lim_{x\to\infty} e^{\frac{s}{a}x} \neq 0.$$

$$(7) \Rightarrow C_2 = g/s^3$$

Thus $(6) \Rightarrow U(x,s) = \frac{g}{s^3} e^{-\frac{s}{a}x} - \frac{g}{s^3}$

Taking L^{-1} on both sides, we get,

$$L^{-1}\{U(x,s)\} = g\, L^{-1}\left\{\frac{1}{s^3} e^{-\frac{x}{a}s}\right\} - g\, L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$\Rightarrow u(x,t) = g\, H\left(t-\frac{x}{a}\right) f\left(t-\frac{x}{a}\right) - g\frac{t^2}{2!} \quad \because L\{t^2\} = \frac{2!}{s^3}$$

where $f(t) = L^{-1}\{F(s)\} = L^{-1}\{1/s^3\}$.

$$\Rightarrow f(t) = \frac{1}{2} t^2. \Rightarrow f\left(t-\frac{x}{a}\right) = \frac{1}{2}\left(t-\frac{x}{a}\right)^2$$

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Special.

Example 4. 2010, 2017,

Solve the problem using Laplace transform method.

$$u_{xx} = u_{tt}, \quad 0<x<1, \quad t>0 \qquad \text{—— (1)}$$

$$u(0,t) = 0, \quad u(1,t) = 0 \qquad \text{—— (2)}$$

$$u(x,0) = 5\sin \pi x ; \quad u_t(x,0) = -5\sin \pi x \qquad \text{—— (3)}$$

Solution.

Taking Laplace transform of (1) and (2), we get

$$(1) \Rightarrow L\left\{\frac{\partial^2}{\partial x^2} u(x,t)\right\} = L\left\{\frac{\partial^2}{\partial t^2} u(x,t)\right\}.$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) = s^2 U(x,s) - su(x,0) - u_t(x,0).$$

$$\Rightarrow \frac{d^2}{dx^2} U(x,s) = s^2 U(x,s) - 5 \cdot 5\sin \pi x + 5\sin \pi x \quad \because \text{by (3)}$$

$$\Rightarrow D^2 U(x,s) - s^2 U(x,s) = -5\, 5\sin \pi x + 5\sin \pi x.$$

The characteristic equation is

$$D^2 - s^2 = 0 \quad \Rightarrow D^2 = s^2$$

$$\Rightarrow D = \pm s.$$

$$\Rightarrow U_c(x,s) = C_1 e^{sx} + C_2 e^{-sx} \qquad \text{—— (4)}$$

Now we find the particular integral as follows,

$$U_p(x,s) = \frac{-5\, 5\sin \pi x + 5\sin \pi x}{D^2 - s^2}$$

$$= \frac{-s \sin \pi x}{D^2 - s^2} + \frac{\sin \pi x}{D^2 - s^2}$$

$$= \frac{-s \sin \pi x}{-\pi^2 - s^2} + \frac{\sin \pi x}{-\pi^2 - s^2}$$

Replace D^2 by $-(\text{coefficient of } x)^2$

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$$= \frac{s \sin\bar{\lambda}x}{\bar{\lambda}^2+s^2} - \frac{\sin\bar{\lambda}x}{\bar{\lambda}^2+s^2}$$

$$\Rightarrow U_p(x,s) = \frac{\sin\bar{\lambda}x}{s^2+\bar{\lambda}^2}(s-1).$$

The general solution is

$$U(x,s) = U_c(x,s) + U_p(x,s)$$

$$\Rightarrow U(x,s) = C_1 e^{sx} + C_2 e^{-sx} + \frac{\sin\bar{\lambda}x}{s^2+\bar{\lambda}^2}(s-1) \quad \text{---(5)}$$

(2) $\Rightarrow L\{U(0,t)\} = 0, \quad L\{U(1,t)\} = 0$

$\Rightarrow U(0,s) = 0, \quad U(1,s) = 0.$

(5) $\Rightarrow U(0,s) = C_1 + C_2 = 0$

$\Rightarrow C_1 = -C_2.$

and $U(1,s) = C_1 e^{s} + C_2 e^{-s} = 0 \quad \because \sin\bar{\lambda} = 0.$

$\Rightarrow -C_2 e^{s} + C_2 e^{-s} = 0$

$\Rightarrow C_2 (e^{-s} - e^{s}) = 0$

$\Rightarrow C_2 = 0 \quad \because e^{-s} - e^{s} \neq 0.$

$\Rightarrow C_1 = 0.$

Thus

(5) $\Rightarrow U(x,s) = \frac{(s-1)}{s^2+\bar{\lambda}^2} \sin\bar{\lambda}x.$

Taking L^{-1} on both sides, we get.

$$L^{-1}\{U(x,s)\} = \sin\bar{\lambda}x\, L^{-1}\left\{\frac{s}{s^2+\bar{\lambda}^2}\right\} - \frac{1}{\bar{\lambda}}\sin\bar{\lambda}x\, L^{-1}\left\{\frac{\bar{\lambda}}{s^2+\bar{\lambda}^2}\right\}$$

$$= \sin\bar{\lambda}x\{\cos\bar{\lambda}t\} - \frac{1}{\bar{\lambda}}\sin\bar{\lambda}x\{\sin\bar{\lambda}t\}$$

$$\Rightarrow U(x,t) = \sin\bar{\lambda}x\left\{\cos\bar{\lambda}t - \frac{1}{\bar{\lambda}}\sin\bar{\lambda}t\right\}.$$

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Example 5. 2018/A, 2018/S

Verify that $\mathcal{L}^{-1}\{\tan^{-1}\frac{a}{s}\} = \frac{1}{t}\sin at$.

Solution.

Given that to show

$$\mathcal{L}^{-1}\{\tan^{-1}\frac{a}{s}\} = \frac{1}{t}\sin at$$

$$\Rightarrow \mathcal{L}\{\frac{1}{t}\cdot \sin at\} = \tan^{-1} a/s \quad \text{——} (1)$$

$$L.H.S = \mathcal{L}\{\frac{\sin at}{t}\}.$$

Since we know that

$$\mathcal{L}\{\frac{f(t)}{t}\} = \int_{u=s}^{\infty} F(u)\,du$$

Therefore, $$\mathcal{L}\{\frac{\sin at}{t}\} = \int_{u=s}^{\infty} \frac{a}{u^2+a^2}\,du \quad \text{——} (2)$$

Note that

$$\frac{d}{du}(\tan^{-1}\frac{a}{u}) = \frac{1}{1+a^2/u^2}\left(\frac{-a}{u^2}\right) = \frac{u^2}{u^2+a^2}\left(-a/u^2\right)$$

$$= \frac{-a}{u^2+a^2}.$$

Thus equation (2) becomes

$$\mathcal{L}\{\frac{\sin at}{t}\} = -\int_{u=s}^{\infty} \frac{d}{du}(\tan^{-1}\frac{a}{u})\,du$$

$$= -\int_{u=s}^{\infty} d(\tan^{-1}\frac{a}{u})$$

$$= -\tan^{-1}\frac{a}{u}\Big|_{u=s}^{\infty}$$

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$$= -\left[\tan^{-1}(0) - \tan^{-1}(a/s)\right] = +\tan^{-1} a/s$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sin at}{t}\right\} = \tan^{-1}(a/s)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\tan^{-1}\frac{a}{s}\right\} = \frac{1}{t}\sin at \quad \text{as required.}$$

A/2018

Q. Find Fourier transform of $xe^{-\alpha x^2}$.

Sol:

By definition

$$\mathcal{F}\{xe^{-\alpha x^2}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} xe^{-\alpha x^2} dx.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{ikx-\alpha x^2} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-\alpha\left(x^2 - \frac{ikx}{\alpha}\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-\alpha\left(x^2 - 2x\frac{ik}{2\alpha} + \frac{i^2k^2}{4\alpha^2} - \frac{i^2k^2}{4\alpha^2}\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-\alpha\left(x - \frac{ik}{2\alpha}\right)^2 - k^2/4\alpha} dx$$

$$= \frac{e^{-k^2/4\alpha}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-\alpha\left(x - \frac{ik}{2\alpha}\right)^2} dx$$

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Put $\sqrt{\alpha}\left(x - \frac{ik}{2\alpha}\right) = t$

Then $\sqrt{\alpha}\, dx = dt$

and $x - \frac{ik}{2\alpha} = \frac{t}{\sqrt{\alpha}}$

$\Rightarrow x = \frac{t}{\sqrt{\alpha}} + \frac{ik}{2\alpha}$

Thus,

$$\mathcal{F}\{x e^{-\alpha x^2}\} = \frac{1}{\sqrt{2\pi}} e^{-k^2/4\alpha} \int_{-\infty}^{\infty} \left(\frac{t}{\sqrt{\alpha}} + \frac{ik}{2\alpha}\right) e^{-t^2} \frac{dt}{\sqrt{\alpha}}$$

$$= \frac{1}{\sqrt{2\pi\alpha}} e^{-k^2/4\alpha} \int_{-\infty}^{\infty} \left(\frac{t}{\sqrt{\alpha}} + \frac{ik}{2\alpha}\right) e^{-t^2} dt$$

$$= \frac{1}{\sqrt{2\pi}\,\alpha} e^{-k^2/4\alpha} \int_{-\infty}^{\infty} t e^{-t^2} dt$$

$$+ \frac{ik}{\sqrt{2\pi}\,(2\alpha)^{3/2}} e^{-k^2/4\alpha} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= +\frac{1}{2\alpha\sqrt{2\pi}} e^{-k^2/4\alpha} \int_{-\infty}^{\infty} e^{-t^2} (+2t)\, dt$$

$$+ \frac{ik}{\sqrt{\pi}\,(2\alpha)^{3/2}} e^{-k^2/4\alpha} \sqrt{\pi} \quad \because \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

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Put $t^2 = u$
$\Rightarrow 2t\,dt = du$

$$= +\frac{1}{2\alpha\sqrt{2\pi}} \cdot e^{-K^2/4\alpha} \int_{-\infty}^{\infty} e^{-u}\,du$$

$$+\frac{iK}{\sqrt[3]{2\alpha}}\, e^{-K^2/4\alpha}$$

$$= \frac{iK}{\sqrt[3]{2\alpha}}\, e^{-K^2/4\alpha}$$

نوٹ:-
$\int_{-\infty}^{\infty} e^{-u}\,du$
$= 2(0)$ odd function
$= 0$

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Fourier Transformation

Fourier Transform.

If $f(x)$ is an integrable function over an interval $(-\infty, \infty)$. Then the Fourier transform of $f(x)$ is defined by

$$F\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx.$$

Here e^{ikx} is kernel of Fourier transform.

Note.

If the function $f(x)$ or $F(k)$ is continuous over the interval $(-\infty, +\infty)$ and bounded then Fourier transform and inverse Fourier transform exist.

Sufficient Condition for the existence of Fourier Transform.

If the function $f(x)$ is absolutely integrable i.e. the integral $\int_{-\infty}^{\infty} |f(x)|\, dx$ exists, then the Fourier transform exists.

Linearity of F and F^{-1} Operators.

The operator F and F^{-1} are linear.

i.e. $F\{C_1 f(x) + C_2 f_2(x)\} = C_1 F\{f_1(x)\} + C_2 F\{f_2(x)\}$.

and $F^{-1}\{C_1 f(x) + C_2 f_2(x)\} = C_1 F\{f_1(x)\} + C_2 F\{f_2(x)\}$

Example 1. 2016,

Find the Fourier transform of Gaussian function.

$$f(x) = Ne^{-\alpha x^2}$$

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where N and α are constants, and $\alpha > 0$.

Solution.

Given that

$$f(x) = Ne^{-\alpha x^2} \quad \text{where } N \text{ and } \alpha \text{ are constants.}$$

Taking Fourier transform, we get,

$$F\{f(x)\} = F\{Ne^{-\alpha x^2}\}.$$

$$\Rightarrow F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} Ne^{-\alpha x^2} dx.$$

$$\Rightarrow F(k) = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha x^2 + ikx} dx. \quad \text{——— (1)}$$

Note that

$$-\alpha x^2 + ikx = -\alpha\left(x^2 - \frac{ik}{\alpha}x\right)$$

$$= -\alpha\left(x^2 - 2x\frac{ik}{2\alpha} + \left(\frac{ik}{2\alpha}\right)^2\right) + \alpha\left(\frac{ik}{2\alpha}\right)^2$$

$$= -\alpha\left(x - \frac{ik}{2\alpha}\right)^2 - \frac{k^2}{4\alpha^2}\alpha.$$

$$\Rightarrow -\alpha x^2 + ikx = -\alpha\left(x - \frac{ik}{2\alpha}\right)^2 - \frac{k^2}{4\alpha}$$

Now equation (1) becomes

$$F(k) = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left(x - \frac{ik}{2\alpha}\right)^2 - k^2/4\alpha} dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left(x - \frac{ik}{2\alpha}\right)^2} e^{-k^2/4\alpha} dx$$

Put $\alpha\left(x - \frac{ik}{2\alpha}\right)^2 = y^2$

$$\Rightarrow \sqrt{\alpha}\left(x - \frac{ik}{2\alpha}\right) = y$$

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$\Rightarrow \sqrt{\alpha}\,dx = dy$; $x \to \infty \Rightarrow y \to \infty$

$\Rightarrow dx = \frac{dy}{\sqrt{\alpha}}$ $x \to -\infty \Rightarrow y \to -\infty$

So $F(k) = \frac{N}{\sqrt{2\pi}} e^{-k^2/4\alpha} \int_{-\infty}^{\infty} e^{-y^2} \frac{dy}{\sqrt{\alpha}}$.

$= \frac{N}{\sqrt{2\pi}\cdot\sqrt{\alpha}} e^{-k^2/4\alpha} \int_{-\infty}^{\infty} e^{-y^2} dy$.

By Gaussian integral.

$\because \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$

$= \frac{N}{\sqrt{2\pi\alpha}} e^{-k^2/4\alpha} \cdot \sqrt{\pi}$

$\Rightarrow F(k) = \frac{N}{\sqrt{2\alpha}} e^{-\frac{k^2}{4\alpha}}$

Example 2. 2017,

Find the Fourier transform of $g(x) = \frac{a}{x^2+a^2}$; $a>0$.

Solution.

The function $g(x)$ as well as its derivative are continuous over the interval $(-\infty, +\infty)$, and the integral $\int_{-\infty}^{\infty} g(x)dx$ is absolutely integrable. Therefore the Fourier transform of the given function must exist.

$$F\{g(x)\} = G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \frac{a}{x^2+a^2} dx.$$

$$= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+a^2} dx.$$

$$\Rightarrow G(k) = \frac{a}{\sqrt{2\pi}} \oint_C \frac{e^{ikz}}{z^2+a^2} dz \quad \text{---(1)}$$

{By contour integration.}

(1) has simple poles at $z = \pm ai$

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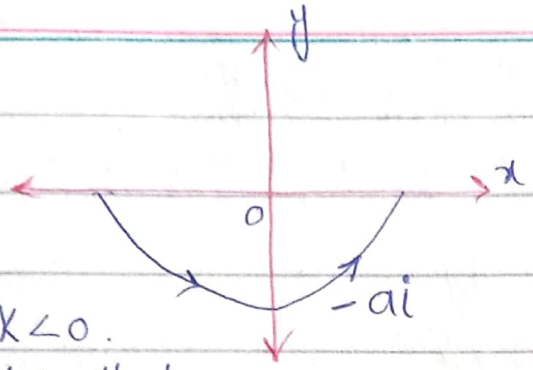
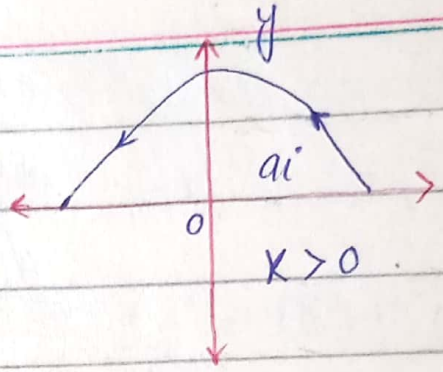


$K<0$.



Note that

$$e^{ikz} = e^{ik(x+iy)} = e^{ikx-ky} = e^{ikx}\cdot e^{-ky}.$$

$\Rightarrow e^{ikz} \to 0$ when $y \to \infty$ for $K>0$.

and $e^{ikz} \to 0$ when $y \to -\infty$ for $K<0$.

Now equation ① becomes

$$G(K) = \frac{a}{\sqrt{2\pi}} 2\pi i \sum_{i=1}^{2} R_i \longrightarrow ②$$

When $K>0$.

$$R_1 = \lim_{z\to ai}\left\{ e^{ikz}\frac{(z-ia)}{z^2+a^2}\right\} = \lim_{z\to ai}\left\{ e^{ikz}\frac{1}{z+ai}\right\}.$$

$$= \frac{e^{ik(ia)}}{ia+ia} = \frac{e^{-ka}}{2ia}$$

When $K<0$.

$$R_2 = \lim_{z\to -ai}\left\{ (z+ia)\frac{e^{ikz}}{z^2+a^2}\right\}$$

$$= \lim_{z\to -ai}\left\{ e^{ikz}\frac{1}{z-ia}\right\}$$

$$= \frac{e^{ik(-ai)}}{-ia-ia} = \frac{e^{ka}}{-2ai}$$

$$\text{So } \sum_{i=1}^{2} R_i = \frac{e^{-ka}}{2ia} + \frac{e^{ka}}{-2ai}$$

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$$\Rightarrow \sum_{i=1}^{2} R_i = \frac{e^{-ka}}{2ia} - \frac{e^{ka}}{2ia}$$

Thus equation (2) becomes

$$G(k) = \frac{a}{\sqrt{2\pi}} 2\pi i \left\{ \frac{e^{-ka}}{2ia} - \frac{e^{ka}}{2ia} \right\}$$

$$= \frac{a}{\sqrt{2\pi}} 2\pi i \left\{ \frac{e^{-ka} - e^{+ka}}{2ia} \right\}$$

$$= -\sqrt{\frac{\pi}{2}} \left\{ e^{+ka} - e^{-ka} \right\}$$

$$= -\sqrt{\frac{\pi}{2}}\, 2 \left\{ \frac{e^{ka} - e^{-ka}}{2} \right\}$$

$$\Rightarrow G(k) = -\sqrt{\frac{\pi}{2}}\, 2 \sinh(ka)$$

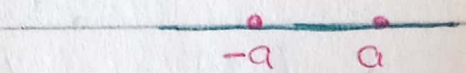
$$= -\sqrt{2\pi} \sinh(ka).$$

Example 3.

Find the Fourier transform of the box function.

$$f(x) = \begin{cases} 1, & |x| \le a \quad a > 0 \\ 0, & |x| > a \end{cases}$$

Solution.



$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-a} e^{ikx} f(x)\, dx + \int_{-a}^{a} e^{ikx} f(x)\, dx + \int_{a}^{\infty} e^{ikx} f(x)\, dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-a} e^{ikx} (0)\, dx + \int_{-a}^{a} e^{ikx} (1)\, dx + \int_{a}^{\infty} e^{ikx} (0)\, dx \right\}$$

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$$\Rightarrow F(k) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{ikx}\, dx = \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{ikx}}{ik} \right\}_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left\{ \frac{e^{ika} - e^{-ika}}{ik} \right\}.$$

$$= \frac{2}{k\sqrt{2\pi}} \left\{ \frac{e^{ika} - e^{-ika}}{2i} \right\}.$$

$$= \frac{1}{k}\sqrt{\frac{2}{\pi}}\, \sin(ka)$$

$$\Rightarrow F(k) = \sqrt{\frac{2}{\pi}}\, \frac{\sin(ka)}{k}.$$

Properties of Fourier Transformation.

① Conjugation Property.

If $f(x)$ is real, then $F(-k) = \overline{F(k)}$.

Proof.

By definition of Fourier transform

$$F\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx. \quad \text{——} \ (1)$$

Taking conjugate, we get

$$\overline{F(k)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, \overline{f(x)}\, dx$$

$$\Rightarrow \overline{F(k)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx. \quad \text{——} \ (2)$$

$\because f(x)$ is real
so $\overline{f(x)} = f(x)$

From equation (1), we can write,

$$F(-k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx \quad \text{——} \ (3)$$

$-k$ کو k سے بدل دیں

Comparing (2) and (3), we have.

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$F(-k) = \overline{F(k)}$.

(2) Real and Complex values of the Fourier Transform. 2014.

(a) If $f(x)$ is real and even, then $F(k)$ is real.

(b) If $f(x)$ is real and odd, then $F(k)$ is pure imaginary. (real part zero).

(c) If $f(x)$ is complex, then $\mathcal{F}\{\overline{f(-x)}\} = \overline{F(k)}$.

Proof.

(a)

Given that $f(x)$ is real and even.

So $\overline{f(x)} = f(x)$ ——— (1)

$f(-x) = f(x)$ ——— (2)

By definition

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx$$

$$\Rightarrow F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(-x)\, dx \quad \because \text{using (2)}$$

Put $-x = x^*$ $\quad x \to -\infty \Rightarrow x^* \to \infty$

$-dx = dx^*$ $\quad x \to \infty \Rightarrow x^* \to -\infty$

Then $F(k) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{ik(-x^*)} f(x^*)(-dx^*)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx^*} f(x^*)\, dx^*$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx \quad \because \text{Definite integral or, variable change}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{e^{ikx}}\, \overline{f(x)}\, dx = \overline{F(k)} \quad \because \text{using (1)}$$

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$\Rightarrow F(k) = \overline{F(k)}$

$\Rightarrow F(k)$ is real.

(b)

Given that $f(x)$ is real and odd.

Therefore, $\overline{f(x)} = f(x)$ ——— (1)

$f(-x) = -f(x)$ ——— (2)

By definition

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \{-f(-x)\}\,dx \quad \text{using (2).}$$

$$\Rightarrow F(k) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(-x)\,dx$$

Put $-x = x^*$ $\quad x \to \infty, \Rightarrow x^* \to -\infty$

$-dx = dx^*$ $\quad x \to -\infty \Rightarrow x^* \to \infty.$

Then $F(k) = \frac{-1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{+ik(-x^*)} f(x^*)(-dx^*).$

$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx^*} f(x^*)\,dx^*$$

$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx.$$

$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{e^{ikx}}\, \overline{f(x)}\,dx \quad \text{: using (1)}$$

$$= -\overline{F(k)}$$

$\Rightarrow F(k) = -\overline{F(k)} \Rightarrow F(k)$ is pure imaginary.

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(C)

By definition

$$F\{\overline{f(-x)}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, \overline{f(-x)}\, dx$$

Put $-x = x^* \Rightarrow -dx = dx^*$

$x \to -\infty \Rightarrow x^* \to \infty$

$x \to +\infty \Rightarrow x^* \to -\infty$.

Then $F\{\overline{f(-x)}\} = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{ik(-x^*)}\, \overline{f(x^*)}(-dx^*)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx^*}\, \overline{f(x^*)}\, dx^*$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, \overline{f(x)}\, dx.$$

$\Rightarrow F\{\overline{f(-x)}\} = \overline{F(k)}$.

③ Attenuation Property:

$$F\{e^{ax} f(x)\} = F(k - ai).$$

Proof.

By definition

$$F\{e^{ax} f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, e^{ax} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(ik+a)x} f(x)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(k + a/i)x} f(x)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(k - ai)x} f(x)\, dx.$$

$$= F(k - ai).$$

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(4) Shifting Properties

(a) $F\{f(x-a)\} = e^{ika} F(k)$

(b) $F\{e^{iax} f(x)\} = F(k+a)$.

Proof.

(a)

$$L.H.S = F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x-a)\,dx.$$

Put $x - a = x^* \Rightarrow dx = dx^*$

Then $F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x^*+a)} f(x^*)\,dx^*$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx^*} \cdot e^{ika} f(x^*)\,dx^*$$

$$= \frac{e^{ika}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx^*} f(x^*)\,dx^*$$

$$= e^{ika}\left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx\right\}$$

$$= e^{ika} F(k) = R.H.S.$$

(b)

$$L.H.S = F\{e^{iax} f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} e^{ikx} f(x)\,dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(k+a)x} f(x)\,dx.$$

$$= F(k+a).$$

$$= R.H.S.$$

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(5) Scaling Property. 2010, 2011,

If c is a non zero constant, then

$$F\{f(cx)\} = \frac{1}{|c|} F(k/c).$$

Proof:

Let $c > 0$, then

$$F\{f(cx)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(cx)\, dx$$

Put $cx = x^* \Rightarrow x = \frac{x^*}{c} \Rightarrow dx = \frac{1}{c} dx^*$.

Then $F\{f(cx)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik\left(\frac{x^*}{c}\right)} f(x^*) \frac{dx^*}{c}$

$$= \frac{1}{c} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\left(\frac{k}{c}\right)x^*} f(x^*)\, dx^*$$

$$= \frac{1}{c} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\left(\frac{k}{c}\right)x} f(x)\, dx \right\}.$$

$$\Rightarrow F\{f(ax)\} = \frac{1}{c} F(k/c) \quad ; c > 0 \quad \text{——— (1)}$$

Let $c < 0$, then

$$F\{f(cx)\} = -\frac{1}{c} F(k/c) \quad \text{——— (2)}$$

Combining (1) and (2), we get,

$$F\{f(cx)\} = \frac{1}{|c|} F(k/c).$$

(6) Modulation Property of Fourier Transform.

(i) $F\{\cos ax\, f(x)\} = \frac{1}{2}[F(k+a) + F(k-a)]$

(ii) $F\{\sin ax\, f(x)\} = \frac{1}{2i}[F(k+a) - F(k-a)]$.

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Proof.

(i) $L.H.S = \mathcal{F}\{\cos ax\, f(x)\} = \mathcal{F}\left\{\frac{e^{iax}+e^{-iax}}{2} f(x)\right\}$.

Since $\mathcal{F}$ is linear operator.

Therefore, $\mathcal{F}\{\cos ax\, f(x)\} = \frac{1}{2}\mathcal{F}\{e^{iax} f(x)\} + \frac{1}{2}\mathcal{F}\{e^{-iax} f(x)\}$

using Shifting Prop.

$$= \frac{1}{2}F(k+a) + \frac{1}{2}F(k-a)$$

$$= \frac{1}{2}\left[F(k+a) + F(k-a)\right]$$

(ii) $\mathcal{F}\{\sin ax\, f(x)\} = \mathcal{F}\left\{\frac{e^{iax}-e^{-iax}}{2i} f(x)\right\}$

$$= \frac{1}{2i}\left[F(k+a) - F(k-a)\right].$$

Special

V.imp (اس بار پیپر میں لازمی)

State and prove boundedness and Continuity of the Fourier transform.

If $f(x)$ is piecewise smooth and absolutely integrable on the interval $(-\infty, \infty)$, then its Fourier transform $F(k)$ is bounded and continuous.

Proof.

By definition of Fourier transform

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

Taking absolute value, we have,

$$|F(k)| = \left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)\,dx\right|$$

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$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{ikx} f(x)|\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{ikx}|\, |f(x)|\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\, dx \qquad \because |e^{ikx}| = 1.$$

$$\Rightarrow |F(k)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\, dx \qquad \text{——— (1)}$$

Since $f(x)$ is absolutely integrable on the interval $(-\infty, +\infty)$ so we can take

$$\int_{-\infty}^{\infty} |f(x)|\, dx = J \quad \text{(say)}$$

where J is a finite +ve numbes

(1) $\Rightarrow |F(k)| \leq \frac{1}{\sqrt{2\pi}} J$

This shows that $F(k)$ is bounded.

Now we show the continuity of $F(k)$ as follows.

$$F(k+h) - F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(k+h)x} f(x)\, dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{e^{ikx} \cdot e^{ihx} - e^{ikx}\} f(x)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} (e^{ihx} - 1) f(x)\, dx.$$

$$= I(k, h) \quad \text{(say)}.$$

Taking $\lim_{h \to 0}$, we get,

$$\lim_{h \to 0} [F(k+h) - F(k)] = \lim_{h \to 0} I(k, h) \qquad \text{——— (2)}$$

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Now $|I(k,h)| = \left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}(e^{ihx}-1)f(x)dx\right|$

$\leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|e^{ikx}(e^{ihx}-1)f(x)|dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|e^{ikx}||e^{ihx}-1||f(x)|dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|e^{ihx}-1||f(x)|dx \quad \because |e^{ikx}|=1$

$\Rightarrow |I(k,h)| \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|e^{ihx}-1||f(x)|dx$ ——— (3)

Note that

$|e^{ihx}-1| = |\cos(hx)+i\sin(hx)-1|$

$= |\cos(hx)-1+i\sin(hx)|$

$= \sqrt{(\cos(hx)-1)^2+\sin^2(hx)}$

$= \sqrt{\cos^2(hx)+1-2\cos(hx)+\sin^2(hx)}$

$= \sqrt{1+1-2\cos(hx)} = \sqrt{2-2\cos(hx)}$

$= \sqrt{2(1-\cos(hx))} = \sqrt{2\left(2\sin^2\frac{hx}{2}\right)}$

$= \sqrt{4\sin^2\left(\frac{hx}{2}\right)} = 2\sin\left(\frac{hx}{2}\right)$

$\leq 2 \quad \because \sin\left(\frac{hx}{2}\right) < 1$

$\Rightarrow |e^{ihx}-1| \leq 2$

Now equation (3) becomes

$|I(k,h)| \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|e^{ihx}-1||f(x)|dx$

$\leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2|f(x)|dx.$

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$$\Rightarrow |I(k,h)| \leq \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} |f(x)|\,dx$$

$$\Rightarrow |I(k,h)| \leq \sqrt{\frac{2}{\pi}}\, J \quad \text{where } J = \int_{-\infty}^{\infty} |f(x)|\,dx.$$

This shows that $I(k,h)$ is uniformly convergent.

Hence equation (2) implies that

$$\lim_{h\to 0}[F(k+h)-F(k)] = \lim_{h\to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}(e^{ihx}-1) f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \lim_{h\to 0}(e^{ihx}-1) f(x)\,dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}(1-1) f(x)\,dx$$

$$\Rightarrow \lim_{h\to 0}[F(k+h)-F(k)] = 0$$

Therefore $F(k)$ is continuous.

Special Theorem (Riemann-Lebesque theorem):

If $f(x)$ is piecewise smooth and absolutely integrable function, then $\lim_{|k|\to\infty} |F(k)| = 0$.

Proof.

By definition of Fourier transform

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx.$$

Integrating R.H.S by parts, we get,

$$F(k) = \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{ikx}}{ik} f(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{e^{ikx}}{ik} f'(x)\,dx \right\}.$$

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taking absolute value, we have,

$$|F(k)| = \left| \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{ikx} f(x)}{ik} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{e^{ikx} f'(x)}{ik} dx \right\} \right|$$

$$\leq \frac{1}{\sqrt{2\pi}} \left\{ \lim_{x\to\infty} \left| \frac{e^{ikx} f(x)}{ik} \right| - \lim_{x\to-\infty} \left| \frac{e^{ikx} f(x)}{ik} \right| \right.$$

absolute value of sum ≤ sum of absolute values

$$\left. + \int_{-\infty}^{\infty} \left| \frac{e^{ikx} f'(x)}{ik} \right| dx \right\}.$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \lim_{x\to\infty} \frac{|e^{ikx}||f(x)|}{|ik|} - \lim_{x\to-\infty} \frac{|e^{ikx}||f(x)|}{|ik|} \right.$$

$$\left. + \int_{-\infty}^{\infty} \frac{|e^{ikx}||f'(x)|}{|ik|} dx \right\}.$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \lim_{x\to\infty} \frac{|f(x)|}{|k|} - \lim_{x\to-\infty} \frac{|f(x)|}{|k|} + \int_{-\infty}^{\infty} \frac{|f'(x)|}{|k|} dx \right\}$$

as $|e^{ikx}| = 1$ and $|i| = 1$.

$$\Rightarrow |F(k)| \leq \frac{1}{|k|\sqrt{2\pi}} \left\{ \lim_{x\to\infty} |f(x)| - \lim_{x\to-\infty} |f(x)| + \int_{-\infty}^{\infty} |f'(x)| dx \right\}.$$

Since $f(x)$ is absolutely integrable,

$$\lim_{x\to\pm\infty} |f(x)| = 0.$$

so $$|F(k)| \leq \frac{1}{|k|\sqrt{2\pi}} \int_{-\infty}^{\infty} |f'(x)| dx. \quad \text{——— (1)}$$

Since $f(x)$ is piecewise smooth so $f'(x)$ is piecewise continuous and therefore

$$\int_{-\infty}^{\infty} |f'(x)| dx = J \text{ (say)}$$

a finite +ve number

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① $\Rightarrow |F(k)| \leq \frac{1}{|k|\sqrt{2\pi}} J$

Taking $\lim_{|k|\to\infty}$ on both sides, we get.

$$\lim_{|k|\to\infty} |F(k)| \leq \lim_{|k|\to\infty} \frac{1}{|k|\sqrt{2\pi}} J$$

$\Rightarrow \lim_{|k|\to\infty} |F(k)| \leq 0$

$\Rightarrow \lim_{|k|\to\infty} |F(k)| = 0$ ∵ absolute value can never be ~~negative~~.

$|F(k)| \geq 0$

$\lim_{|k|\to\infty} |F(k)| \geq 0$

and $\lim_{|k|\to\infty} |F(k)| \leq 0$

So $\lim_{|k|\to\infty} |F(k)| = 0$

Fourier Transforms of derivatives.

$$\mathcal{F}\{f'(x)\} = (-ik)F(k)$$

where $f(x)$ supposed to be zero as $x \to \pm\infty$.

$$\mathcal{F}\{f''(x)\} = (-ik)^2 F(k)$$

where $f(x)$, $f'(x)$ supposed to be zero as $x \to \pm\infty$

$$\mathcal{F}\{f^{(n)}(x)\} = (-ik)^n F(k)$$

where $f(x), f'(x), f''(x), \cdots, f^{(n-1)}(x)$ supposed to be zero as $x \to \pm\infty$.

Proof.

By definition of ~~Laplace~~ Fourier transform

$$\mathcal{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f'(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ e^{ikx} f(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (ik) e^{ikx} f(x)\,dx \right\}.$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \lim_{x\to\infty} e^{ikx} f(x) - \lim_{x\to-\infty} e^{ikx} f(x) \right\} - \frac{ik}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

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Since $f(x) \to 0$ as $x \to \pm\infty$.

Therefore, $F\{f'(x)\} = \frac{1}{\sqrt{2\pi}}(0) - \frac{ik}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$

$= -ik\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)\,dx\right\}$.

$\Rightarrow F\{f'(x)\} = (-ik)\,F(k)$ ——— (1)

and

$F\{f''(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f''(x)\,dx$.

$= \frac{1}{\sqrt{2\pi}}\left\{e^{ikx} f'(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\, e^{ikx}(ik)\,dx\right\}$.

$= \frac{1}{\sqrt{2\pi}}\left\{\lim_{x\to\infty} e^{ikx} f'(x) - \lim_{x\to-\infty} e^{ikx} f'(x)\right\}$

$-\frac{1}{\sqrt{2\pi}}(ik)\int_{-\infty}^{\infty} e^{ikx} f'(x)\,dx\Big\}$.

$= \frac{1}{\sqrt{2\pi}}(0) - \frac{1}{\sqrt{2\pi}}(ik)\int_{-\infty}^{\infty} e^{ikx} f'(x)\,dx$ $\because f'(x) \to 0$ as $x \to \pm\infty$

$= (-ik)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f'(x)\,dx\right\}$.

$= -ik\left[F\{f'(x)\}\right]$

$\Rightarrow F\{f''(x)\} = (-ik)\left[(-ik)\,F(k)\right\}$ using (1).

$= (-ik)^2 F(k)$.

Continuing in this way, we get,

$F\{f^{(n)}(x)\} = (-ik)^n F(k)$.

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Fourier Transform of $x^n f(x)$.

Let n be a +ve integer and $f(x)$ a pairwise continuous function on the interval $[-l, l]$ for every +ve l. Suppose that $\int_{-\infty}^{\infty} |x^n f(x)|\,dx$ converges.

Then $$F\{x^n f(x)\} = (i)^{-n} \frac{d^n}{dk^n} F(k).$$

Proof.

By definition

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

Diff. w.r.t. k, we get,

$$\frac{d}{dk} F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} (ix) f(x)\,dx.$$

Again diff. w.r.t. k, we have.

$$\frac{d^2}{dk^2} F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} (ix)^2 f(x)\,dx.$$

Continuing in this way, we get,

$$\frac{d^n}{dk^n} F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} (ix)^n f(x)\,dx.$$

$$= \frac{i^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} x^n f(x)\,dx.$$

$$\Rightarrow \frac{1}{i^n} \frac{d^n}{dk^n} F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} x^n f(x)\,dx \quad \text{——— (1)}$$

Also by def. we can write,

$$F\{f(x) x^n\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} x^n f(x)\,dx \quad \text{——— (2)}$$

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Comparing ① and ②, we have,

$\mathcal{F}\{x^n f(x)\} = (i)^{-n} \frac{d^n}{dk^n} F(k)$. as required.

Fourier Transform of an integral.

Suppose that $f(x)$ is piecewise continuous on $(-\infty, +\infty)$ and that $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$. Also let $F(0) = 0$, with $F(k) = \mathcal{F}\{f(x)\}$.

Then

$$\mathcal{F}\left\{\int_{-\infty}^{x} f(x^*)\,dx^*\right\} = -\frac{1}{ik} F(k).$$

Proof.

Let $g(x) = \int_{-\infty}^{x} f(x^*)\,dx^*$ ——— ①

Then by the Leibniz rule

$$g'(x) = f(x) \quad \text{——— ②}$$

whenever $f(x)$ is continuous.

① $\Rightarrow g(-\infty) = \int_{-\infty}^{-\infty} f(x^*)\,dx^*$

upper and lower limits are same.

$\Rightarrow g(-\infty) = 0$

By definition

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

$$\Rightarrow F(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\,dx \quad \text{——— ③} \because k = 0$$

Taking $\lim_{x \to \infty}$ of equation ①, we get.

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \int_{-\infty}^{x} f(x^*)\,dx^*$$

$$= \int_{-\infty}^{\infty} f(x^*)\,dx^*$$

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$$= \sqrt{2\pi}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,dx\right\}$$

In Definite integral variable can be change

$$\Rightarrow \lim_{x\to\infty} g(x) = \sqrt{2\pi}\,F(0)$$ using ③

As $F(k) = \mathcal{F}\{f(x)\}$

$= \mathcal{F}\{g'(x)\}$ using ②

$\Rightarrow F(k) = \mathcal{F}\{g'(x)\}$ ——— ④

Also

$$\mathcal{F}\{g'(x)\} = (-ik)\,\mathcal{F}\{g(x)\}$$

So ④ $\Rightarrow F(k) = (-ik)\,\mathcal{F}\{g(x)\}$.

$$\Rightarrow \mathcal{F}\{g(x)\} = \frac{-1}{ik}F(k)$$

$$\Rightarrow \mathcal{F}\left\{\int_{-\infty}^{x} f(x^*)\,dx^*\right\} = \frac{-1}{ik}F(k)$$ using ①.

Convolution. 2010, 2011,

If $f(x)$ and $g(x)$ are functions of x defined over the interval $(-\infty, \infty)$ then their convolution denoted by $f * g$, is defined by

$$f * g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,g(x-u)\,du.$$

The following results hold.

$$f * g = g * f$$

$$f * (g+h) = (f * g) + (f * h)$$

$$f * (g * h) = (f * g) * h$$

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Convolution Theorem. 2014,

If $F(k)$ and $G(k)$ are Fourier transforms of $f(x)$ and $g(x)$, then

$$\mathcal{F}\{f * g\} = F(k)\,G(k).$$

Proof.

By definition of inverse Fourier transform

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)G(k)\,dk. \quad \text{——— (1)}$$

Also by definition of Fourier transform

$$G(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx^*} g(x^*)\,dx^*$$

so

$$(1) \Rightarrow \mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx^*} g(x^*)\,dx^*\right\} dk$$

Changing order of integration, we get,

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} e^{-ik(x-x^*)} F(k)\,dk\right\} g(x^*)\,dx^*$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ik(x-x^*)} F(k)\,dk\right\} g(x^*)\,dx^* \quad \text{——— (2)}$$

By definition

$$f(x-x^*) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ik(x-x^*)} F(k)\,dk.$$

Thus equation (2) becomes.

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{f(x-x^*)\}\,g(x^*)\,dx^*.$$

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$= f * g$

$\Rightarrow \bar{F}^{-1}\{F(k)G(k)\} = f * g$

$\Rightarrow F\{f * g\} = F(k)\,G(k)$ as required.

Parseval Theorem. 2010, 2017/A

$$\int_{-\infty}^{\infty} F(k)\,\overline{F(k)}\,dk = \int_{-\infty}^{\infty} f(x)\,\overline{f(x)}\,dx$$ 1st P. theorem

Or $$\int_{-\infty}^{\infty} |F(k)|^2\,dk = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

$\{\because z\bar{z} = |z|^2\}$

Proof.

By convolution theorem

$$F\{f * g\} = F(k)G(k).$$

$$\Rightarrow f * g = \bar{F}^{-1}\{F(k)G(k)\}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,g(x-u)\,du = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,F(k)G(k)\,dk$$

Putting $x = 0$ on both sides, we get,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,g(-u)\,du = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)G(k)\,dk.$$

Or $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,g(-x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)G(k)\,dk \quad \text{——— (1)}$$

Eq (1) is also 2nd P theorem

Let

$$g(-x) = \overline{f(x)}$$

$$\Rightarrow g(x) = \overline{f(-x)}$$ replacing x by $-x$.

Then

$$F\{g(x)\} = F\{\overline{f(-x)}\} \quad \text{——— (2)}$$

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Since we know that if $f(x)$ is complex, then $\mathcal{F}\{\overline{f(-x)}\} = \overline{F(k)}$.

Therefore, equation (2) implies

$$G(k) = \overline{F(k)}.$$

Now equation (1) becomes

$$\int_{-\infty}^{\infty} f(x)\,\overline{f(x)}\,dx = \int_{-\infty}^{\infty} F(k)\,\overline{F(k)}\,dk.$$

$$\Rightarrow \int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(k)|^2\,dk.$$

The Fourier integral Theorem.

If $f(x)$ is real function defined over an interval $(-\infty,\infty)$, and the integral $\int_{-\infty}^{\infty} f(x)\,dx$ is absolutely convergent, then

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(x^*)\cos K(x-x^*)\,dx^*\,dK.$$

Proof.

Since the integral $\int_{-\infty}^{\infty} f(x)\,dx$ is absolutely convergent so its Fourier transform and the inverse both exist.

Therefore, $$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)\,dk$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{-ikx} F(k)\,dk + \int_0^{\infty} e^{-ikx} F(k)\,dk\right]$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\left[I_1 + \int_0^{\infty} e^{-ikx} F(k)\,dk\right] \quad \text{---------- (1)}$$

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where $I_1 = \int_{-\infty}^{0} e^{-ikx} F(k)\,dk$

Putting $-k = k^* \Rightarrow +dk = -dk^*$, in I_1 we have

$K \to -\infty \quad k \to 0$
$\Rightarrow k^* \to \infty \quad k^* \to 0$

$$I_1 = \int_{\infty}^{0} e^{ik^*x} F(-k^*)(-dk^*)$$

$$= \int_{0}^{\infty} e^{ik^*x} F(-k^*)\,dk^*$$

Since $f(x)$ is real, therefore, by Conjugation property

$$F(-k^*) = \overline{F(k^*)}.$$

So $I_1 = \int_0^{\infty} e^{ik^*x}\, \overline{F(k^*)}\, dk^*$

or $I_1 = \int_0^{\infty} e^{ikx}\, \overline{F(k)}\, dk$ $\quad \because k^*$ is dummy variable.

Now equation ① becomes

$$f(x) = \frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{ikx}\,\overline{F(k)}\,dk + \int_0^{\infty} e^{-ikx} F(k)\,dk\right]$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left[e^{ikx}\,\overline{F(k)} + e^{-ikx} F(k)\right] dk \quad \text{—— ②}$$

Also by definition of Fourier transform

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx^*} f(x^*)\,dx^*$$

Xing both sides by e^{-ikx}, we get,

$$e^{-ikx} F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, e^{ikx^*} f(x^*)\,dx^*$$

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$$\Rightarrow e^{-ikx} F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik(x-x^*)} f(x^*) dx^* \longrightarrow (3)$$

Taking complex conjugate of both sides, we get,

$$\overline{e^{ikx} F(k)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{e^{ik(x-x^*)} f(x^*)}\, dx^*.$$

$$\Rightarrow e^{ikx} \overline{F(k)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x-x^*)} f(x^*) dx^* \quad \because f(x) \text{ is real}$$

——— (4)

Adding equation (3) and (4), we get,

$$e^{-ikx} F(k) + e^{ikx} \overline{F(k)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik(x-x^*)} f(x^*) dx^* + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x-x^*)} f(x^*) dx^*.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{ e^{-ik(x-x^*)} + e^{ik(x-x^*)} \} f(x^*) dx^*.$$

——— (5)

Note that

$$e^{-ik(x-x^*)} + e^{ik(x-x^*)} = \cos k(x-x^*) - i \sin k(x-x^*) + \cos k(x-x^*) + i \sin k(x-x^*)$$

$$= 2 \cos k(x-x^*)$$

Now equation (5) becomes

$$e^{-ikx} F(k) + e^{ikx} \overline{F(k)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2 \cos k(x-x^*) f(x^*) dx^*$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x^*) \cos k(x-x^*) dx^*$$

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Thus equation (2) becomes

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^\infty f(x^*) \cos k(x - x^*)\, dx^*\, dk$$

$$\Rightarrow f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(x^*) \cos k(x - x^*)\, dx^*\, dk.$$

as required

Fourier Sine and Cosine Transforms.

If a function $f(x)$ is defined over the interval $[0, \infty)$, then its Fourier sine transform $F_s(k)$ and Fourier cosine transform $F_c(k)$ are defined as

$$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx\, dx$$

and

$$F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos kx\, dx.$$

The inverse relations are

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(k) \sin kx\, dk$$

and

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(k) \cos kx\, dk.$$

Fourier Sine and Cosine transforms of derivatives.

To calculate Fourier sine and cosine transforms of first order derivative, we assume that $f(x)$ is real and $|f(x)| \to 0$ as $x \to \infty$. Then

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$$F_c\{f'(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos kx\, dx.$$

Integrating by parts, we have,

$$F_c\{f'(x)\} = \sqrt{\frac{2}{\pi}} \left[\cos kx \cdot f(x) \Big|_0^\infty - \int_0^\infty f(x)(-\sin kx)k\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\lim_{x\to\infty} f(x)\cos kx - \lim_{x\to 0} f(x)\cos kx + k\int_0^\infty f(x)\sin kx\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - f(0) + k\int_0^\infty f(x)\sin kx\, dx \right] \quad \because f(x)\to 0 \text{ as } x\to\infty$$

$$= -\sqrt{\frac{2}{\pi}} f(0) + k\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin kx\, dx.$$

$$\Rightarrow F_c\{f'(x)\} = -\sqrt{\frac{2}{\pi}} f(0) + k\, F_s(k).$$

and

$$F_s\{f'(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x)\sin kx\, dx.$$

Integrating by parts, we get,

$$F_s\{f'(x)\} = \sqrt{\frac{2}{\pi}} \left\{ \sin kx\, f(x) \Big|_0^\infty - \int_0^\infty f(x)\cos kx \cdot k\, dx \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \lim_{x\to\infty} f(x)\sin kx - \lim_{x\to 0} f(x)\sin kx - k\int_0^\infty f(x)\cos kx\, dx \right\}$$

$$= -k\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos kx\, dx.$$

$$\Rightarrow F_s\{f'(x)\} = -k\, F_c(k).$$

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To calculate Fourier sine and cosine transforms of 2nd order derivates, we further assume that $|f'(x)| \to 0$ as $x \to \infty$. Then

2010/2014 $F_c\{f''(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f''(x) \cos kx \, dx.$

$= \sqrt{\frac{2}{\pi}} \left[\cos kx \, f'(x) \Big|_0^\infty - \int_0^\infty f'(x) (-\sin kx) k \, dx \right]$

$= \sqrt{\frac{2}{\pi}} \left[\lim_{x\to\infty} f'(x) \cos kx - \lim_{x\to 0} f'(x) \cos kx + k \int_0^\infty f'(x) \sin kx \, dx \right]$

$= \sqrt{\frac{2}{\pi}} \left[0 - f'(0) + k \int_0^\infty f'(x) \sin kx \, dx \right]$

$= -\sqrt{\frac{2}{\pi}} f'(0) + k \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin kx \, dx.$

$= -\sqrt{\frac{2}{\pi}} f'(0) + k \, F_s\{f'(x)\}.$

$= -\sqrt{\frac{2}{\pi}} f'(0) + k \{ -k F_c(k) \}$

$\Rightarrow F_c\{f''(x)\} = -\sqrt{\frac{2}{\pi}} f'(0) - k^2 F_c(k).$

and.

$F_s\{f''(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f''(x) \sin kx \, dx.$

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$= \sqrt{\frac{2}{\pi}} \left[\sin kx \, f'(x) \Big|_0^\infty - k \int_0^\infty f'(x) \cos kx \, dx \right]$

$= \sqrt{\frac{2}{\pi}} \left[\lim_{x\to\infty} f'(x) \sin kx - \lim_{x\to 0} f'(0) \sin kx - k \int_0^\infty f'(x) \cos kx \, dx \right]$

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$$= -k\sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos kx\,dx$$

$$= -k\,F_c\{f'(x)\}$$

$$= -k\left\{-\sqrt{\frac{2}{\pi}}f(0) + k\,F_s(k)\right\}.$$

$$F_s\{f''(x)\} = k\sqrt{\frac{2}{\pi}}f(0) - k^2F_s(k).$$

Special Example 1.

Calculate Fourier sine transform of the function $f(x) = e^{-x}\cos x$.

Solution.

By definition

$$F_s\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin kx\,dx$$

$$\Rightarrow F_s\{e^{-x}\cos x\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos x\sin kx\,dx.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin kx\cos x\,dx.$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty 2e^{-x}\sin kx\cos x\,dx.$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\{\sin(k+1)x + \sin(k-1)x\}dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin(k+1)x\,dx + \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin(k-1)x\,dx$$

$$\Rightarrow F_s\{e^{-x}\cos x\} = \frac{1}{\sqrt{2\pi}}[I_1 + I_2] \quad \text{——— (1)}$$

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Where $I_1 = \int_0^\infty e^{-x} \sin(K+1)x\,dx$

and $I_2 = \int_0^\infty e^{-x} \sin(K-1)x\,dx$.

Since we know that

$$\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\{a\sin bx - b\cos bx\}$$

Therefore,

$$I_1 = \int_0^\infty e^{-x} \sin(K+1)x\,dx$$

$$= \left[\frac{e^{-x}}{(-1)^2+(K+1)^2}\{(-1)\sin(K+1)x - (K+1)\cos(K+1)x\}\right]_0^\infty$$

$$= \left[\frac{e^{-x}}{1+K^2+1+2K}\{-\sin(K+1)x - (K+1)\cos(K+1)x\}\right]_0^\infty$$

$$= 0 - \frac{1}{K^2+2K+2}\{-0-(K+1)(1)\}.$$

$$\Rightarrow I_1 = \frac{K+1}{K^2+2K+2} \quad \text{---------- (2)}$$

and $$I_2 = \left[\frac{e^{-x}}{(-1)^2+(K-1)^2}\{(-1)\sin(K-1)x - (K-1)\cos(K-1)x\}\right]_0^\infty$$

$$= 0 - \frac{1}{1+K^2+1-2K}\{-0-(K-1)(1)\}$$

$$\Rightarrow I_2 = \frac{K-1}{K^2-2K+2} \quad \text{---------- (3)}$$

Adding (2) and (3), we get,

$$I_1 + I_2 = \frac{K+1}{K^2+2K+2} + \frac{K-1}{K^2-2K+2}$$

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$$= \frac{(K+1)(K^2-2K+2) + (K-1)(K^2+2K+2)}{(K^2+2K+2)(K^2-2K+2)}$$

$$= \frac{K^3-2K^2+2K+K^2-2K+2+K^3+2K^2+2K-K^2-2K-2}{K^4-2K^3+2K^2+2K^3-4K^2+4K+2K^2-4K+4}$$

$$= \frac{2K^3}{K^4+4}$$

$$\Rightarrow I_1 + I_2 = \frac{2K^3}{K^4+4}$$

Now equation ① becomes,

$$F_s\{e^{-x}\cos x\} = \frac{1}{\sqrt{2\pi}}\left\{\frac{2K^3}{K^4+4}\right\}.$$

$$= \sqrt{\frac{2}{\pi}}\ \frac{K^3}{K^4+4}$$

Example 2.

Calculate Fourier sine transform of the function

$$f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$$

Solution.

By definition

$$F_s\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\sin kx\,dx.$$

$$= \sqrt{\frac{2}{\pi}}\left\{\int_0^{\pi} f(x)\sin kx\,dx + \int_{\pi}^{\infty} f(x)\sin kx\,dx\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{\int_0^{\pi} \sin x \sin kx\,dx + \int_{\pi}^{\infty} (0)\sin kx\,dx\right\}.$$

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$$= \sqrt{\frac{2}{\pi}} \int_0^{\pi} \sin kx \sin x \, dx = -\frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^{\pi} (-2\sin kx \sin x)\, dx$$

$$= -\frac{1}{\sqrt{2\pi}} \int_0^{\pi} \{\cos(k+1)x - \cos(k-1)x\}\, dx.$$

$$= \frac{-1}{\sqrt{2\pi}} \left\{ \int_0^{\pi} \cos(k+1)x\, dx - \int_0^{\pi} \cos(k-1)x\, dx \right\}.$$

$$\Rightarrow \bar{F}_s\{f(x)\} = \frac{-1}{\sqrt{2\pi}} \{I_1 - I_2\} \quad \text{---------- (1)}$$

where

$$I_1 = \int_0^{\pi} \cos(k+1)x\, dx \quad \text{and} \quad I_2 = \int_0^{\pi} \cos(k-1)x\, dx$$

Now,

$$I_1 = \left. \frac{\sin(k+1)x}{k+1} \right|_0^{\pi} = \frac{\sin(k+1)\pi}{k+1} - 0.$$

$$= \frac{1}{k+1} \{\sin k\pi \cos\pi + \cos(k\pi) \sin\pi)\}$$

$$\Rightarrow I_1 = -\frac{\sin k\pi}{k+1} \quad \text{---------- (2)}$$

and $I_2 = \left. \frac{\sin(k-1)}{k-1} \right|_0^{\pi} = \frac{\sin(k-1)\pi}{k-1} - 0.$

$$= \frac{1}{k-1} \{\sin k\pi \cos\pi - \cos k\pi \sin\pi\}$$

$$\Rightarrow I_2 = -\frac{\sin k\pi}{k-1} \quad \text{---------- (3)}$$

Subtracting (3) from (2), we get.

$$I_1 - I_2 = -\frac{\sin k\pi}{k+1} + \frac{\sin k\pi}{k-1}$$

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$$= \sin k\pi \left\{ \frac{-(k-1)+(k+1)}{(k+1)(k-1)} \right\}.$$

$$= \sin k\pi \left\{ \frac{-k+1+k+1}{k^2-1} \right\}$$

$$\Rightarrow I_1 - I_2 = \frac{2\sin k\pi}{k^2-1}$$

Now equation (1) becomes

$$\tilde{F}_s\{f(x)\} = \frac{-1}{\sqrt{2\pi}} \left\{ \frac{2\sin k\pi}{k^2-1} \right\}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin k\pi}{1-k^2}$$

Example 3.

Show that

(a) $\tilde{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{2ak}{(a^2+k^2)^2}$

(b) $\tilde{F}_c\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{a^2-k^2}{(a^2+k^2)^2}$.

Solution.

(a)

By definition

$$\tilde{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty xe^{-ax} \sin kx\, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x\,(e^{-ax}\sin kx)\, dx.$$

Integrating by parts, we get,

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$$= \sqrt{\frac{2}{\pi}}\left[x \cdot \frac{e^{-ax}}{(-a)^2+K^2}(-a\sin Kx - K\cos Kx)\Big|_0^\infty - \int_0^\infty \frac{e^{-ax}}{a^2+K^2}(-a\sin Kx - K\cos Kx)\,dx \right].$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - 0 + \int_0^\infty \frac{e^{-ax}}{a^2+K^2}(a\sin Kx + K\cos Kx)\,dx \right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+K^2}\int_0^\infty e^{-ax}\sin Kx\,dx + \frac{K}{a^2+K^2}\int_0^\infty e^{-ax}\cos Kx\,dx \right]$$

$$\Rightarrow F_s\{x e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+K^2} I_1 + \frac{K}{a^2+K^2} I_2 \right] \quad \text{——— (1)}$$

where $I_1 = \int_0^\infty e^{-ax}\sin Kx\,dx$ and $I_2 = \int_0^\infty e^{-ax}\cos Kx\,dx$.

Now, $I_1 = \frac{e^{-ax}}{a^2+K^2}(-a\sin Kx - K\cos Kx)\Big|_0^\infty$

$$= 0 - \frac{1}{a^2+K^2}(-0 - K)$$

$$\Rightarrow I_1 = \frac{K}{a^2+K^2} \quad \text{——— (2)}$$

and $I_2 = \frac{e^{-ax}}{a^2+K^2}(-a\cos Kx + K\sin Kx)\Big|_0^\infty$

$$= 0 - \frac{1}{a^2+K^2}(-a + 0)$$

$$\Rightarrow I_2 = \frac{a}{a^2+K^2} \quad \text{——— (3)}$$

Subsituting (2) and (3) in (1), we get,

$$F_s\{x e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+K^2}\cdot\frac{K}{a^2+K^2} + \frac{K}{a^2+K^2}\cdot\frac{a}{a^2+K^2} \right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{aK}{(a^2+K^2)^2} + \frac{aK}{(a^2+K^2)^2} \right]$$

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$$F_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{2ak}{(a^2+k^2)^2}\right]$$ as required.

(b).

By definition

$$F_c\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-ax}\cos kx\,dx.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty x\left(e^{-ax}\cos kx\right)dx$$

Integrating by parts, we get,

$$F_c\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[x\cdot\frac{e^{-ax}}{a^2+k^2}(-a\cos kx + k\sin kx)\Big|_0^\infty - \int_0^\infty \frac{e^{-ax}}{a^2+k^2}(-a\cos kx + k\sin kx)\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[0-0+\int_0^\infty \frac{e^{-ax}}{a^2+k^2}(a\cos kx + k\sin kx)\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+k^2}\int_0^\infty e^{-ax}\cos kx\,dx + \frac{k}{a^2+k^2}\int_0^\infty e^{-ax}\sin kx\,dx\right]$$

$$\Rightarrow F_c\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+k^2}I_1 + \frac{k}{a^2+k^2}I_2\right] \quad \text{---- (1)}$$

where $I_1 = \int_0^\infty e^{-ax}\cos kx\,dx$ and $I_2 = \int_0^\infty e^{-ax}\sin kx\,dx$

$$\Rightarrow I_1 = \frac{a}{a^2+k^2} \quad \text{and} \quad I_2 = \frac{k}{a^2+k^2} \quad \text{---- (2)}$$

Substituting (2) in (1), we get

$$F_c\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+k^2}\cdot\frac{a}{a^2+k^2} + \frac{k}{a^2+k^2}\cdot\frac{k}{a^2+k^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a^2}{(a^2+k^2)^2} + \frac{k^2}{(a^2+k^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{a^2+k^2}{(a^2+k^2)^2}$$ as required.

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Example 4.

Determine $F_c\{x^{\alpha-1}\}$ and $F_s\{x^{\alpha-1}\}$.

Solution.

By definition

$$F_c\{x^{\alpha-1}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{\alpha-1} \cos kx\, dx \quad \text{——— (1)}$$

and

$$F_s\{x^{\alpha-1}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{\alpha-1} \sin kx\, dx \quad \text{——— (2)}$$

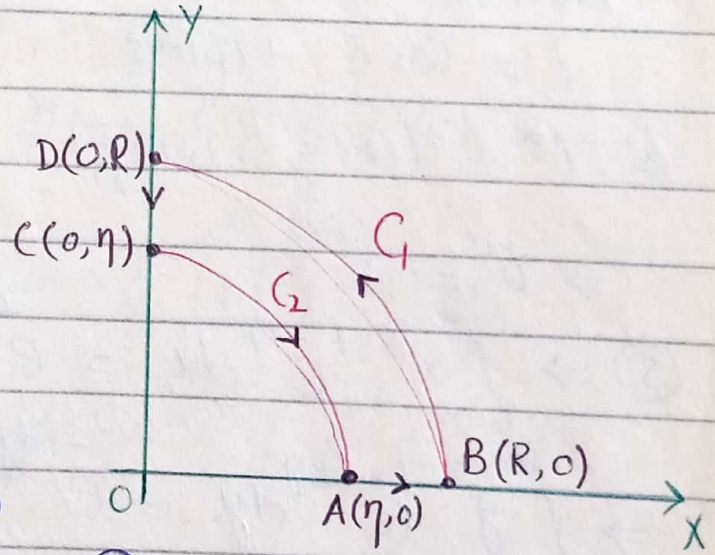
To calculate the integrals on R.H.S of (1) and (2), we define

$$f(z) = z^{\alpha-1} e^{-kz}, \quad 0 < \alpha < 1.$$

If this is analytic in C, then by Cauchy's theorem

$$\oint_C f(z)\,dz = 0$$

$$\Rightarrow \int_{BD} f(z)dz + \int_{DC} f(z)dz + \int_{CA} f(z)dz + \int_{AB} f(z)dz = 0 \quad \longrightarrow (3)$$



Since $f(z) = z^{\alpha-1} e^{-kz}$

Therefore, $f(x+iy) = (x+iy)^{\alpha-1} e^{-k(x+iy)}$

Along DC: $f(z) = (iy)^{\alpha-1} e^{-k(iy)}$ $\because x=0$

Along AB: $f(z) = x^{\alpha-1} e^{-kx}$ $\because y=0$.

$$(3) \Rightarrow \int_{C_1} f(z)dz + \int_R^\eta (iy)^{\alpha-1} e^{-iky}\,(idy) + \int_{C_2} f(z)dz + \int_\eta^R x^{\alpha-1} e^{-kx}\,dx = 0$$

$z = x+iy$
$z = iy$ when $x=0$
$dz = idy$
$z = x$ when $y=0$
$dz = dx$

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$$\Rightarrow \int_{C_1} f(z)dz + i^{\alpha}\int_{R}^{\eta} y^{\alpha-1}e^{-iky}dy + \int_{C_2} f(z)dz + \int_{\eta}^{R} x^{\alpha-1}e^{-kx}dx = 0 \longrightarrow (4)$$

If the limit $\eta \to 0, R \to \infty$, then $\int_{C_1} f(z)dz = 0$ and $\int_{C_2} f(z)dz = 0$.

So (4) $\Rightarrow i^{\alpha}\int_{\eta=0}^{R=\infty} y^{\alpha-1}e^{-iky}dy + \int_{0}^{\infty} x^{\alpha-1}e^{-kx}dx = 0$

$$\Rightarrow \int_0^{\infty} x^{\alpha-1}e^{-kx}dx = -i^{\alpha}\int_{\infty}^{0} y^{\alpha-1}e^{-iky}dy$$

$$= i^{\alpha}\int_0^{\infty} y^{\alpha-1}e^{-iky}dy. \longrightarrow (5)$$

Note that

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$\Rightarrow i^{\alpha} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{\alpha} = \left(e^{\frac{i\pi}{2}}\right)^{\alpha}$$

$$\Rightarrow i^{\alpha} = e^{\frac{i\alpha\pi}{2}}$$

$$(5) \Rightarrow \int_0^{\infty} x^{\alpha-1}e^{-kx}dx = e^{\frac{i\alpha\pi}{2}}\int_0^{\infty} y^{\alpha-1}e^{-iky}dy$$

$$\Rightarrow \int_0^{\infty} y^{\alpha-1}e^{-iky}dy = e^{-\frac{i\alpha\pi}{2}}\int_0^{\infty} x^{\alpha-1}e^{-kx}dx$$

$$\Rightarrow \int_0^{\infty} y^{\alpha-1}\{\cos(ky) - i\sin(ky)\}dy = \left(\cos\alpha\frac{\pi}{2} - i\sin\alpha\frac{\pi}{2}\right)\int_0^{\infty} x^{\alpha-1}e^{-kx}dx.$$

$$\Rightarrow \int_0^{\infty} y^{\alpha-1}\cos kydy - i\int_0^{\infty} y^{\alpha-1}\sin kydy = \cos\alpha\frac{\pi}{2}\int_0^{\infty} x^{\alpha-1}e^{-kx}dx - i\sin\alpha\frac{\pi}{2}\int_0^{\infty} x^{\alpha-1}e^{-kx}dx$$

Equating real and imaginary parts, we get

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$$\int_0^\infty y^{\alpha-1}\cos ky\,dy = \cos\alpha\tfrac{\pi}{2}\int_0^\infty x^{\alpha-1}e^{-kx}\,dx \quad \text{——} (6)$$

and

$$\int_0^\infty y^{\alpha-1}\sin ky\,dy = \sin\alpha\tfrac{\pi}{2}\int_0^\infty x^{\alpha-1}e^{-kx}\,dx \quad \text{——} (7)$$

Put $kx=t \Rightarrow x=t/k \Rightarrow dx=\frac{dt}{k}$

Then (6) $\Rightarrow \int_0^\infty y^{\alpha-1}\cos ky\,dy = \cos\alpha\tfrac{\pi}{2}\int_0^\infty (\tfrac{t}{k})^{\alpha-1}e^{-t}\frac{dt}{k}$

$$= \cos\alpha\tfrac{\pi}{2}\int_0^\infty \frac{t^{\alpha-1}}{k^{\alpha-1}}e^{-t}\frac{dt}{k}$$

$$= \frac{1}{k^\alpha}\cos\alpha\tfrac{\pi}{2}\int_0^\infty e^{-t}t^{\alpha-1}\,dt$$

{y is dummy index}

$$\Rightarrow \int_0^\infty x^{\alpha-1}\cos kx\,dx = \frac{1}{k^\alpha}\cos\alpha\tfrac{\pi}{2}\,\Gamma(\alpha) \quad \text{——} (8)$$

and

$$(7) \Rightarrow \int_0^\infty y^{\alpha-1}\sin ky\,dy = \sin\alpha\tfrac{\pi}{2}\int_0^\infty (\tfrac{t}{k})^{\alpha-1}e^{-t}\frac{dt}{k}$$

$$= \frac{1}{k^\alpha}\sin\alpha\tfrac{\pi}{2}\int_0^\infty e^{-t}t^{\alpha-1}\,dt.$$

$$\Rightarrow \int_0^\infty x^{\alpha-1}\sin kx\,dx = \frac{1}{k^\alpha}\sin\alpha\tfrac{\pi}{2}\,\Gamma(\alpha). \quad \text{——} (9)$$

{y is dummy index}

Substituting (8) and (9) in (1) and (2) respectively, we get,

$$(1) \Rightarrow \hat{f}_c\{x^{\alpha-1}\} = \sqrt{\frac{2}{\pi}}\,\Gamma(\alpha)\frac{\cos(\alpha\frac{\pi}{2})}{k^\alpha}$$

$$(2) \Rightarrow \hat{f}_s\{x^{\alpha-1}\} = \sqrt{\frac{2}{\pi}}\,\Gamma(\alpha)\frac{\sin(\alpha\frac{\pi}{2})}{k^\alpha}.$$

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Note.

If the problem contains B.C. of the form $u(0,t) = \text{constant}$, then we use sine transform. On the other hand if the problem contains a B.C. of the form $u_x(0,t) = \text{constant}$, then we use cosine transform.

Example 1.

Solve the potential equation for the potential $u(x,y)$ in the semi-infinite strip $0<x<c, y>0$ that satisfies the following conditions

$$u(0,y) = 0, \quad u_y(x,0) = 0 \quad ; u_x(c^*, y) = f(y)$$

Solution.

The potential equation is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0<x<c, \quad y>0 \qquad \text{---------- (1)}$$

and the given conditions are

$$u(0,y) = 0 \quad , u_y(x,0) = 0, \quad u_x(c^*, y) = f(y) \qquad \text{---------- (2)}$$

Taking Fourier cosine transform of (1), we get,

$$F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} + F_c\left\{\frac{\partial^2 u}{\partial y^2}\right\} = 0 \qquad \text{---------- (3)}$$

Consider $F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u(x,y)}{\partial x^2} \cos ky \, dy$

$$= \sqrt{\frac{2}{\pi}} \frac{\partial^2}{\partial x^2} \int_0^\infty u(x,y) \cos ky \, dy.$$

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$$= \frac{\partial^2}{\partial x^2} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,y) \cos ky \, dy.$$

$$= \frac{\partial^2}{\partial x^2} F_c\{u(x,y)\}.$$

$$\Rightarrow F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2}{dx^2} U_c(x,k) \quad \text{---------- (4)}$$

and $$F_c\left\{\frac{\partial^2 u}{\partial y^2}\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u(x,y)}{\partial y^2} \cos ky \, dy.$$

$$= \sqrt{\frac{2}{\pi}} \left[\cos ky \frac{\partial u}{\partial y} \Big|_0^\infty - \int_0^\infty \frac{\partial u}{\partial y} (-k \sin ky) dy \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\lim_{y \to \infty} \frac{\partial u}{\partial y} \to 0 \cos ky - \lim_{y \to 0} \frac{\partial u(x,y)}{\partial y} \cos ky \right.$$

$$\left. + k \int_0^\infty \frac{\partial u}{\partial y} \sin ky \, dy \right]$$

$$= -\sqrt{\frac{2}{\pi}} \frac{\partial u(x,0)}{\partial y} + k \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial y} \sin ky \, dy$$

$$= -\sqrt{\frac{2}{\pi}} \underbrace{u_y(x,0)}_{\text{given}} \to 0 + k \sqrt{\frac{2}{\pi}} \left[u(x,y) \sin ky \Big|_0^\infty \right.$$

$$\left. - k \int_0^\infty u(x,y) \cos ky \, dy \right]$$

$$= k \sqrt{\frac{2}{\pi}} \left[\lim_{y \to \infty} u(x,y) \to 0 \sin ky - \lim_{y \to 0} u(x,y) \to 0 \sin ky \right.$$

$$\left. - k \int_0^\infty u(x,y) \cos ky \, dy \right]$$

$$= -k^2 \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,y) \cos ky \, dy.$$

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$$= -k^2 F_c\{u(x,y)\}$$

$$\Rightarrow F_c\left\{\frac{\partial^2 u}{\partial y^2}\right\} = -k^2 U_c(x,k) \quad \text{——— (5)}$$

Subsituting (4) and (5) in (3), we get

$$\frac{d^2}{dx^2} U_c(x,k) - k^2 U_c(x,k) = 0$$

The characteristic equation is

$$D^2 - k^2 = 0$$

$$\Rightarrow D = \pm k$$

The general solution is

$$U_c(x,k) = C_1 e^{kx} + C_2 e^{-kx} \quad \text{——— (6)}$$

Taking Fourier Cosine transform of (2), we get,

$$F_c\{u(0,y)\} = 0$$

$$\Rightarrow U_c(0,k) = 0$$

$$(6) \Rightarrow C_1 + C_2 = 0$$

$$\Rightarrow C_1 = -C_2 \quad \text{——— (7)}$$

and

$$F_c\{u_x(c^*,y)\} = F_c\{f(y)\}$$

$$\Rightarrow \frac{d}{dx} U_c(c^*,k) = F_c(k)$$

$$(6) \Rightarrow \frac{d}{dx} U_c(x^*,k) = C_1 k e^{kx} - k C_2 e^{-kx}$$

$$\Rightarrow \frac{d}{dx} U_c(c^*,k) = C_1 k e^{kc^*} - k C_2 e^{-kc^*}$$

$$F_c(k) = C_1 k e^{kc^*} - C_2 k e^{-kc^*}$$

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$$= -C_2 K e^{Kc^*} - C_2 K e^{-Kc^*} \quad \text{using (7)}$$

$$= -C_2 K (e^{Kc^*} + e^{-Kc^*})$$

$$= -2K C_2 \left(\frac{e^{Kc^*} + e^{-Kc^*}}{2}\right)$$

$$\Rightarrow F_c(K) = -2K C_2 \cosh(Kc^*)$$

$$\Rightarrow C_2 = \frac{-F_c(K)}{2K\cosh(Kc^*)}$$

$$(7) \Rightarrow C_1 = \frac{F_c(K)}{2K\cosh(Kc^*)}$$

Now equation (6) becomes

$$U_c(x,K) = \frac{F_c(K)}{2K\cosh(Kc^*)} \{e^{Kx} - e^{-Kx}\}.$$

$$= \frac{F_c(K)}{K\cosh(Kc^*)} \cdot \left(\frac{e^{Kx} - e^{-Kx}}{2}\right)$$

$$= \frac{F_c(K)}{K\cosh(Kc^*)} \cdot \sinh(Kx).$$

Taking inverse Fourier Cosine transform, we get

$$F_c^{-1}\{U_c(x,K)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{F_c(K)}{K\cosh(Kc^*)} \cdot \sinh(Kx)\cos Ky\, dK$$

$$\Rightarrow u(x,y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sinh(Kx)}{K\cosh(Kc^*)} \cos Ky\, F_c(K)\, dK.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sinh(Kx)}{K\cosh(Kc^*)} \cos Ky \sqrt{\frac{2}{\pi}} \int_0^\infty f(y^*)\cos Ky^*\, dy^*\, dK$$

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$$= \frac{2}{\pi} \int_0^\infty \int_0^\infty \frac{\sinh(kx)}{k\cosh(kc^*)} \cos ky\, f(y^*) \cos ky^*\, dy^*\, dk$$

Example 2.

Special

Solve the problem using Fourier transform method.

$$u_t = u_{xx}, \quad x>0, \quad t>0$$

$$u(0,t) = u_0, \quad u(x,0) = 0, \quad u_0 > 0.$$

Solution.

Given that

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \quad \text{——— (1)}$$

$$u(0,t) = u_0, \quad u(x,0) = 0 \quad \text{——— (2)}$$

Taking Fourier Sine transform of (1) and (2), we get

$$(1) \Rightarrow F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = F_s\left\{\frac{\partial u}{\partial t}\right\} \quad \text{——— (3)}$$

Consider

$$F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u(x,t)}{\partial x^2} \sin kx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\partial u(x,t)}{\partial x} \sin kx \Big|_0^\infty - k \int_0^\infty \frac{\partial u}{\partial x} \cos kx\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\lim_{x\to\infty} \frac{\partial u}{\partial x} \sin kx - \lim_{x\to 0} \frac{\partial u(x,t)}{\partial x} \sin kx - k \int_0^\infty \frac{\partial u}{\partial x} \cos kx\, dx \right]$$

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$$= -k\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial x}\cos kx\,dx.$$

$$= -k\sqrt{\frac{2}{\pi}}\left[\cos kx\,u(x,t)\Big|_0^\infty - \int_0^\infty u(x,t)(-k\sin kx)\,dx\right]$$

$$= -k\sqrt{\frac{2}{\pi}}\left[\lim_{x\to 0} u(x,t)\cos kx - \lim_{x\to 0} u(x,t)\cos kx + k\int_0^\infty u(x,t)\sin kx\,dx\right]$$

$$= +k^2\sqrt{\frac{2}{\pi}}\,u(0,t) - k^2\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\sin kx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\,k\,u(0,t) - k^2 F_s\{u(x,t)\}.$$

$$\Rightarrow F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \sqrt{\frac{2}{\pi}}\,k\,u(0,t) - k^2 U_s(x,t) \quad \text{——(4)}$$

and $$F_s\left\{\frac{\partial u(x,t)}{\partial t}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\sin kx\,dx.$$

$$= \frac{\partial}{\partial t}\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\sin kx\,dx.$$

$$= \frac{\partial}{\partial t} F_s\{u(x,t)\}.$$

$$= \frac{\partial}{\partial t} U_s(k,t)$$

$$\Rightarrow F_s\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt} U_s(k,t) \quad \text{——(5)} \quad \because k \text{ is constant.}$$

Subsituting (4) and (5) in (3), we get

$$\frac{d}{dt} U_s(k,t) = \sqrt{\frac{2}{\pi}}\,k\,u(0,t) - k^2 U_s(k,t)$$

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$$\Rightarrow \frac{d}{dt} U_s(k,t) + k^2 U_s(k,t) = \sqrt{\frac{2}{\pi}}\, k\, u_0 \quad : \text{(given)}$$

which is 1st order linear non homogeneous ODE, whose integrating factor is $e^{k^2 t}$ and whose solution is given by

$$e^{k^2 t} U_s(k,t) = \int e^{k^2 t} \sqrt{\frac{2}{\pi}}\, k\, u_0\, dt + C$$

$$= \sqrt{\frac{2}{\pi}}\, k\, u_0 \int e^{k^2 t}\, dt + C$$

$$= \sqrt{\frac{2}{\pi}}\, k\, u_0\, \frac{e^{k^2 t}}{k^2} + C$$

$$\Rightarrow e^{k^2 t} U_s(k,t) = \sqrt{\frac{2}{\pi}}\, k\, u_0\, \frac{e^{k^2 t}}{k^2} + C e^{-k^2 t}$$

$$\Rightarrow U_s(k,t) = \sqrt{\frac{2}{\pi}}\, \frac{u_0}{k} + C e^{-k^2 t} \quad \text{——— (6)}$$

(2) $\Rightarrow F_s\{u(x,0)\} = 0$

$\Rightarrow U_s(k,0) = 0$

(6) $\Rightarrow U_s(k,0) = \sqrt{\frac{2}{\pi}}\, \frac{u_0}{k} + C$

$\Rightarrow 0 = \sqrt{\frac{2}{\pi}}\, \frac{u_0}{k} + C$

$\Rightarrow C = -\sqrt{\frac{2}{\pi}}\, \frac{u_0}{k}$

Now equation (6) becomes

$$U_s(k,t) = \sqrt{\frac{2}{\pi}}\, \frac{u_0}{k} - \sqrt{\frac{2}{\pi}}\, \frac{u_0}{k} e^{-k^2 t}$$

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Taking inverse Fourier sine transform, we get

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{u_0}{k} (1 - e^{-k^2 t}) \sin kx \, dk$$

$$\Rightarrow u(x,t) = \frac{2}{\pi} \frac{u_0}{k} \int_0^\infty (1 - e^{-k^2 t}) \sin kx \, dk.$$

Example 3.

Solve the problem using Fourier transform method.

$$u_t = u_{xx}, \quad x > 0, \quad t > 0$$

$$u_x(0,t) = f(t), \; u(x,0) = 0, \quad x > 0, \; t > 0.$$

Solution.

Given that

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x > 0, \; t > 0 \qquad \text{——— (1)}$$

and $u_x(0,t) = f(t), \; u(x,0) = 0$ ——— (2)

Taking Fourier cosine transform of (1) and (2), we get

$$(1) \Rightarrow F_c\left\{\frac{\partial u}{\partial t}\right\} = F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} \qquad \text{——— (3)}$$

Consider $F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u(x,t)}{\partial x^2} \cos kx \, dx$

$$= \sqrt{\frac{2}{\pi}} \left[\cos kx \, \frac{\partial u}{\partial x} \Big|_0^\infty - \int_0^\infty \frac{\partial u}{\partial x} (-k \sin kx) \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\lim_{x \to \infty} \frac{\partial u}{\partial x} \to 0 - \lim_{x \to 0} \frac{\partial u(x,t)}{\partial x} \cos kx + k \int_0^\infty \frac{\partial u}{\partial x} \sin kx \, dx \right]$$

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$$= +\sqrt{\frac{2}{\pi}}\left[-u_x(0,t) + k\int_0^\infty \frac{\partial u}{\partial x}\sin kx\,dx\right]$$

$$= +\sqrt{\frac{2}{\pi}}\left[-u_x(0,t) + k\left[\sin kx\,u(x,t)\Big|_0^\infty - \int_0^\infty u(x,t)\cos kx\cdot k\,dx\right]\right]$$

$$= +\sqrt{\frac{2}{\pi}}\left[-u_x(0,t) + k\left[\lim_{x\to\infty} u(x,t)\sin kx - \lim_{x\to 0} u(x,t)\sin kx - k\int_0^\infty u(x,t)\cos kx\,dx\right]\right]$$

$$= -\sqrt{\frac{2}{\pi}}\,u_x(0,t) - k^2\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\cos kx\,dx.$$

$$= -\sqrt{\frac{2}{\pi}}\,u_x(0,t) - k^2 F_c\{u(x,t)\}.$$

$$= -\sqrt{\frac{2}{\pi}}\,u_x(0,t) - k^2 U_c(k,t) \quad \text{——— (4)}$$

and

$$F_c\left\{\frac{\partial u}{\partial t}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u(x,t)}{\partial t}\cos kx\,dx.$$

$$= \frac{\partial}{\partial t}\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\cos kx\,dx.$$

$$= \frac{\partial}{\partial t}U_c(k,t)$$

$$\Rightarrow F_c\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}U_c(k,t) \quad \text{——— (5)}$$

Subsituting (4) and (5) in (3), we get,

$$\frac{d}{dt}U_c(k,t) = -\sqrt{\frac{2}{\pi}}\,u_x(0,t) - k^2 U_c(k,t).$$

$$\Rightarrow \frac{d}{dt}U_c(k,t) + k^2 U_c(k,t) = -\sqrt{\frac{2}{\pi}}\,f(t) \text{ given}$$

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which is 1st order linear non homogeneous ODE, whose integrating factor is e^{k^2t} and solution is given by

$$e^{k^2t}\,U_c(k,t) = \int e^{k^2t}\left\{-\sqrt{\frac{2}{\pi}}\,f(t)\right\}dt + c$$

$$= -\sqrt{\frac{2}{\pi}}\int e^{k^2t} f(t)\,dt + c$$

$$\Rightarrow U_c(k,t) = -\sqrt{\frac{2}{\pi}}\,e^{-k^2t}\int_0^t e^{k^2t^*} f(t^*)\,dt^* + c\,e^{-k^2t}$$

$$\Rightarrow U_c(k,t) = -\sqrt{\frac{2}{\pi}}\int_0^t e^{-k^2(t-t^*)} f(t^*)\,dt^* + c\,e^{-k^2t} \quad \text{——⑥}$$

② $\Rightarrow \mathcal{F}_c\{u(x,0)\} = 0$

$\Rightarrow U_c(k,0) = 0$

⑥ $\Rightarrow U_c(k,0) = -\sqrt{\frac{2}{\pi}}\int_0^0 e^{-k^2(t-t^*)} f(t^*)\,dt^* + c.$

$\Rightarrow 0 = 0 + c \Rightarrow \boxed{c=0}$

Now equation ⑥ becomes

$$U_c(k,t) = -\sqrt{\frac{2}{\pi}}\int_0^t e^{-k^2(t-t^*)} f(t^*)\,dt^*$$

Taking inverse Fourier cosine transform, we get

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos kx \int_0^t \left\{-\sqrt{\frac{2}{\pi}}\,e^{-k^2(t-t^*)} f(t^*)\,dt^*\right\}dk$$

$$\Rightarrow u(x,t) = -\frac{2}{\pi}\int_0^\infty \cos kx \int_0^t e^{-k^2(t-t^*)} f(t^*)\,dt^*\,dk.$$

//

Example 4.

Solve the problem by means of the Fourier transform method.

$u_t = u_{xx}$, $0 < x < \infty$, $t > 0$.

$u_x(0,t) = 0$; $u(x,0) = f(x)$, $0 < x < \infty$, $t > 0$.

Solution.

Given that

$$u_t = u_{xx} \quad \text{——— (1)}$$

$$u_x(0,t) = 0, \quad u(x,0) = f(x) \quad \text{——— (2)}$$

Taking Fourier cosine transform of (1) and (2), we get

$$(1) \Rightarrow F_c\left\{\frac{\partial u}{\partial t}\right\} = F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} \quad \text{——— (3)}$$

Consider $F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos kx\, dx$

$$= -\sqrt{\frac{2}{\pi}} u_x(0,t) - k^2 U_c(k,t)$$

and

$$\text{——— (4)}$$

$$F_c\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt} U_c(k,t) \quad \text{——— (5)}$$

Substituting (4) and (5) in (3), we get

$$\frac{d}{dt} U_c(k,t) = -\sqrt{\frac{2}{\pi}} u_x(0,t) - k^2 U_c(k,t)$$

$$\Rightarrow \frac{d}{dt} U_c(k,t) = -k^2 U_c(k,t)$$

$$\Rightarrow \frac{d}{dt} U_c(k,t) + k^2 U_c(k,t) = 0.$$

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$\Rightarrow D + k^2 = 0 \quad \because D = d/dt$

$\Rightarrow D = -k^2$

$\Rightarrow U_c(k,t) = C_1 e^{-k^2 t}$ ——— (6)

(2) $\Rightarrow$ $\because F_c\{u(x,0)\} = F_c\{f(x)\}$.

$\Rightarrow U_c(k,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x^*) \cos kx^* dx^*$.

(6) $\Rightarrow$ $U_c(k,0) = C_1$

$\Rightarrow \sqrt{\frac{2}{\pi}} \int_0^\infty f(x^*) \cos kx^* dx^* = C_1$

Putting the value of C_1 in (6), we get,

$U_c(k,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x^*) \cos kx^* e^{-k^2 t} dx^*$.

Taking inverse cosine Fourier transform, we get,

$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-k^2 t} f(x^*) \cos kx^* dx^* \cos kx\, dk$

$\Rightarrow u(x,t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(x^*) \cos kx^* e^{-k^2 t} \cos kx\, dx^* dk$.

Example 5. 2016,

Solve the problem by using the Fourier transform method

$u_{xx}(x,t) = u_t(x,t)$, $-\infty < x < \infty$, $t > 0$ ——— (1)

$u(x,0) = e^{-\alpha x^2}$, $u(x)$ and $u_x(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$. ——— (2)

Solution.

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Taking Fourier transform of both sides of ①, w.r.t. x, we get,

$$\mathcal{F}\{u_{xx}(x,t)\} = \mathcal{F}\{u_t(x,t)\}.$$

$$\Rightarrow -K^2 U(K,t) = \frac{d}{dt} U(K,t)$$

$$\Rightarrow \frac{d}{dt} U(K,t) \Big/ U(K,t) = -K^2$$

Integrating, we get

$$\ln U(K,t) = -K^2 t + C.$$

$$\Rightarrow U(K,t) = e^{-K^2 t + C}$$

$$\Rightarrow U(K,t) = e^{-K^2 t} \cdot e^{C}$$

$$= C_1 e^{-K^2 t} \quad \text{——— ③} \quad \because C_1 = e^{C}.$$

$$② \Rightarrow \mathcal{F}\{u(x,0)\} = \mathcal{F}\{e^{-\alpha x^2}\}.$$

$$\Rightarrow U(K,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} \cdot e^{-\alpha x^2} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left(x^2 - \frac{iK}{\alpha}\right)} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left[x^2 - \frac{2iK}{2\alpha} + \left(\frac{iK}{2\alpha}\right)^2 - \left(\frac{iK}{2\alpha}\right)^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left[\left(x - \frac{iK}{2\alpha}\right)^2 + K^2/4\alpha^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left(x - \frac{iK}{2\alpha}\right)^2 - K^2/4\alpha} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left(x - iK/2\alpha\right)^2} e^{-K^2/4\alpha} dx$$

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$$= \frac{1}{\sqrt{2\pi}} e^{-k^2/4\alpha} \int_{-\infty}^{\infty} e^{-[\sqrt{\alpha}(x - ik/2\alpha)]^2} dx.$$

Let $\sqrt{\alpha}(x - ik/2\alpha) = t$ then $\sqrt{\alpha}\,dx = dt$

$$\Rightarrow \mathcal{F}\{e^{-\alpha x^2}\} = \frac{1}{\sqrt{2\pi}} e^{-k^2/4\alpha} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{\sqrt{\alpha}}$$

$$= \frac{1}{\sqrt{2\pi\alpha}} e^{-k^2/4\alpha} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \frac{1}{\sqrt{2\pi\alpha}} e^{-k^2/4\alpha} \cdot \sqrt{\pi} \qquad \because \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

$$\Rightarrow U(k,0) = \frac{1}{\sqrt{2\alpha}} e^{-k^2/4\alpha} \quad \text{——— (4)}$$

(3) $\Rightarrow U(k,0) = C_1$

$$\Rightarrow \frac{1}{\sqrt{2\alpha}} e^{-k^2/4\alpha} = C_1$$

Thus (3) $\Rightarrow U(k,t) = \frac{1}{\sqrt{2\alpha}} e^{-k^2/4\alpha} \cdot e^{-k^2 t}$

$$= \frac{1}{\sqrt{2\alpha}} e^{-k^2\left(\frac{1}{4\alpha} + t\right)}$$

$$\Rightarrow U(k,t) = \frac{1}{\sqrt{2\alpha}} e^{-k^2\beta} \quad \text{where } \beta = t + \frac{1}{4\alpha}$$

Taking Fourier inverse transform, we get

$$\mathcal{F}^{-1}\{U(k,t)\} = \mathcal{F}^{-1}\left\{\frac{1}{\sqrt{2\alpha}} e^{-k^2\beta}\right\}$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left\{\frac{1}{\sqrt{2\alpha}} e^{-k^2\beta}\right\} dk$$

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$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\alpha}} \int_{-\infty}^{\infty} e^{-ikx - k^2\beta} dk.$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\alpha}} I_1 \quad \text{---------- (5)}$$

where $I_1 = \int_{-\infty}^{\infty} e^{-ikx-k^2\beta} dk.$

$$= \int_{-\infty}^{\infty} e^{-\beta\left(\frac{ikx}{\beta} + k^2\right)} dk$$

$$= \int_{-\infty}^{\infty} e^{-\beta\left(k^2 + \frac{2ik}{2\beta}x + \left(\frac{ix}{2\beta}\right)^2 - \left(\frac{ix}{2\beta}\right)^2\right)} dk.$$

$$= \int_{-\infty}^{\infty} e^{-\beta\left[\left(k + \frac{ix}{2\beta}\right)^2 + \frac{x^2}{4\beta^2}\right]} dk$$

$$= \int_{-\infty}^{\infty} e^{-\beta\left(k + \frac{ix}{2\beta}\right)^2 - \frac{x^2}{4\beta}} dk.$$

$$= \int_{-\infty}^{\infty} e^{-\beta\left(k + \frac{ix}{2\beta}\right)^2} e^{-\frac{x^2}{4\beta}} dk.$$

$$= e^{-x^2/4\beta} \int_{-\infty}^{\infty} e^{-\beta\left(k + \frac{ix}{2\beta}\right)^2} dk.$$

$$= e^{-x^2/4\beta} \int_{-\infty}^{\infty} e^{-\left[\sqrt{\beta}\left(k + \frac{ix}{2\beta}\right)^2\right]} dk.$$

Put $\theta = \sqrt{\beta}\left(k + \frac{ix}{2\beta}\right) \Rightarrow d\theta = \sqrt{\beta}\, dk.$

Then $I_1 = e^{-x^2/4\beta} \int_{-\infty}^{\infty} e^{-\theta^2} \frac{d\theta}{\sqrt{\beta}}$

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$$= \frac{1}{\sqrt{\beta}} e^{-x^2/4\beta} \int_{-\infty}^{\infty} e^{-\theta^2} d\theta$$

By Gaussian integral

$$\Rightarrow I_1 = \frac{1}{\sqrt{\beta}} \sqrt{\pi} e^{-x^2/4\beta} = \sqrt{\frac{\pi}{\beta}} e^{-x^2/4\beta} \quad \because \int_{-\infty}^{\infty} e^{-\theta^2} d\theta = \sqrt{\pi}$$

Now equation (5) becomes

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\alpha}} \sqrt{\frac{\pi}{\beta}} e^{-x^2/4\beta}$$

$$= \frac{1}{\sqrt{4\alpha\beta}} e^{-x^2/4\beta}$$

$$= \frac{1}{\sqrt{4\alpha(t+\frac{1}{4\alpha})}} e^{-x^2/4(t+\frac{1}{4\alpha})}$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{1+4\alpha t}} e^{\frac{-x^2\alpha}{1+4\alpha t}}$$

$$\mathcal{F}\{f''(x)\} = (-ix)^2 F(x)$$

Special Example 6.

Solve the problem by means of Fourier transform method.

$$u_t = \alpha^2 u_{xx} \quad -\infty < x < \infty, t > 0 \quad \text{---} (1)$$

$$u(x,0) = f(x), |u(x,0)| < \infty \quad \text{---} (2)$$

Solution.

Taking the Fourier transform of (1), we get

$$\frac{dU(k,t)}{dt} = -\alpha^2 k^2 U(k,t).$$

$$\Rightarrow \frac{dU(k,t)/dt}{U(k,t)} = -\alpha^2 k^2$$

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$\Rightarrow \ln U(k,t) = -\alpha^2 k^2 t + c$: (integrating)

$\Rightarrow U(k,t) = e^{-\alpha^2 k^2 t + c}$

$\Rightarrow U(k,t) = e^{-\alpha^2 k^2 t} \cdot e^c$

$\Rightarrow U(k,t) = C_1 e^{-\alpha^2 k^2 t} \rightarrow (3) \because e^c = C_1$

(2) $\Rightarrow \bar{F}\{u(x,0)\} = \bar{F}\{f(x)\}$

$\Rightarrow U(k,0) = F(k)$

(3) $\Rightarrow U(k,0) = C_1 \Rightarrow \boxed{C_1 = F(k)}$

So

$$U(k,t) = F(k)\, e^{-\alpha^2 k^2 t}$$

Taking inverse Fourier transform, we get

$$\bar{F}^{-1}\{U(k,t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left(F(k)\, e^{-\alpha^2 k^2 t}\right) dk$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx^*} f(x^*)\, dx^*\, e^{-\alpha^2 k^2 t}\, dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \cdot e^{-\alpha^2 k^2 t} \int_{-\infty}^{\infty} e^{ikx^*} f(x^*)\, dx^*\, dk.$$

Changing the order of integration, we get,

$$u(x,t) = \frac{1}{(\sqrt{2\pi})^2} \int_{-\infty}^{\infty} f(x^*) \left[\int_{-\infty}^{\infty} e^{-ikx} \cdot e^{-\alpha^2 k^2 t}\, e^{ikx^*}\, dk\right] dx^*$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x^*) \left[\int_{-\infty}^{\infty} e^{-ik(x-x^*) - \alpha^2 k^2 t}\, dk\right] dx^*$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x^*)\, I(x,x^*)\, dx^* \quad \text{---- (4)}$$

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where $I(x,x^*) = \int_{-\infty}^{\infty} e^{-ik(x-x^*) - \alpha^2 k^2 t}\, dk$.

$= \int_{-\infty}^{\infty} e^{-iku - k^2\beta}\, dk$.

where

$u = x - x^*$ and $\beta = \alpha^2 t$.

$\Rightarrow I(x,x^*) = \sqrt{\frac{\pi}{\beta}}\, e^{-u^2/4\beta}$

Now equation (4) becomes

$u(x,t) = \frac{1}{2\pi}\sqrt{\frac{\pi}{\beta}} \int_{-\infty}^{\infty} f(x^*)\, e^{-u^2/4\beta}\, dx^*$.

By back Subsitutions $u = x - x^*$ and $\beta = \alpha^2 t$.

$u(x,t) = \frac{1}{2\sqrt{\pi\alpha^2 t}} \int_{-\infty}^{\infty} f(x^*)\, e^{-(x-x^*)^2/4\alpha^2 t}$.

Example 7. 2014 (for $a=2$)

Solve by the Fourier transform method

$$u_{xxxx} = \frac{1}{a^2} u_{tt} \qquad \text{——— (1)}$$

where $u(x,0) = f(x)$, $u_t(x,0) = a g'(x)$ ——— (2)

Solution.

Taking Fourier transform of (1), we get

$\mathcal{F}\{u_{xxxx}(x,t)\} = \frac{1}{a^2}\mathcal{F}\{u_{tt}\}$.

$\Rightarrow (-ik)^4 U(k,t) = \frac{1}{a^2}\frac{d^2}{dt^2} U(k,t)$

$\Rightarrow k^4 U(k,t) = \frac{1}{a^2}\frac{d^2}{dt^2} U(k,t)$

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$$\Rightarrow \frac{d^2}{dt^2}U(k,t) - a^2k^4U(k,t) = 0$$

$$\Rightarrow D^2U(k,t) - a^2k^4U(k,t) = 0 \quad \because \frac{d^2}{dt^2} = D^2$$

The charac. equation is

$$D^2 - a^2k^4 = 0$$

$$\Rightarrow D^2 = a^2k^4 \qquad \Rightarrow D = \pm ak^2.$$

The general solution is

$$U(k,t) = C_1 e^{ak^4} + C_2 e^{-ak^2t} \quad \text{——— (3)}$$

Also (2) $\Rightarrow$

$$\mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\}$$

$$\Rightarrow U(k,0) = F(k) \quad \text{——— (4)}$$

and

$$\mathcal{F}\{u_t(x,0)\} = a\mathcal{F}\{g'(x)\}.$$

$$\Rightarrow \frac{d}{dt}U(k,0) = a(-ik)G(k).$$

$$\Rightarrow \frac{d}{dt}U(k,0) = -iakG(k) \quad \text{——— (5)}$$

(3) $\Rightarrow U(k,0) = C_1 + C_2$

(4) $\Rightarrow C_1 + C_2 = F(k) \quad \text{——— (6)}$

Also

$$(3) \Rightarrow \frac{d}{dt}U(k,t) = C_1 e^{ak^2t} \cdot ak^2 - ak^2 C_2 e^{ak^2t}.$$

$$\Rightarrow \frac{d}{dt}U(k,0) = ak^2C_1 - ak^2C_2$$

$$(5) \Rightarrow -ik\,G(k) = k^2(C_1 - C_2)$$

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$$\Rightarrow -i\,G(k) = k(G_1 - G_2)$$

$$\Rightarrow G_1 - G_2 = -\frac{i}{k}G(k) \quad \text{——— (7)}$$

Adding (6) and (7), we get,

$$2G_1 = F(k) - \frac{i}{k}G(k)$$

$$\Rightarrow G_1 = \frac{1}{2}\left[F(k) - \frac{i}{k}G(k)\right] \quad \text{——— (8)}$$

Subtracting (7) from (6), we get

$$2G_2 = F(k) + \frac{i}{k}G(k)$$

$$\Rightarrow G_2 = \frac{1}{2}\left[F(k) + \frac{i}{k}G(k)\right] \quad \text{——— (9)}$$

Subsituting (8) and (9) in (3), we have

$$U(k,t) = \frac{1}{2}\left[F(k) - \frac{i}{k}G(k)\right]e^{ak^2t} + \frac{1}{2}\left[F(k) + \frac{i}{k}G(k)\right]e^{-ak^2t}$$

Taking Fourier inverse transform, we get

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left[\frac{1}{2}F(k)\left(e^{ak^2t} + e^{-ak^2t}\right) - \frac{1}{2}\frac{i}{k}G(k)\left(e^{ak^2t} - e^{-ak^2t}\right)\right]dk$$

$$\Rightarrow u(x,t) = \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left[F(k)\left(e^{ak^2t} + e^{-ak^2t}\right) - \frac{i}{k}G(k)\left(e^{ak^2t} - e^{-ak^2t}\right)\right]dk.$$

required sol.

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See book P297 complete

Special

Example 8.

Solve the following problem by the Fourier transform method.

$$u_{xx}(x,t) = \frac{1}{c^2} u_{tt}(x,t) \quad \text{——— (1)}$$

$$u(x,0) = p(x), \quad u_t(x,0) = q(x) \quad \text{——— (2)}$$

and $u, u_x \to$ as $x \to \pm\infty$.

Solution.

Taking Fourier transform of (1), we get

$$\mathcal{F}\{u_{xx}(x,t)\} = \frac{1}{c^2} \mathcal{F}\{u_{tt}(x,t)\}$$

$$\Leftrightarrow (ik)^2 U(k,t) = \frac{1}{c^2} \frac{d^2}{dt^2} U(k,t).$$

$$\Rightarrow -k^2 U(k,t) = \frac{1}{c^2} \frac{d^2}{dt^2} U(k,t)$$

$$\Rightarrow \frac{d^2}{dt^2} U(k,t) + c^2k^2 U(k,t) = 0.$$

$$\Rightarrow D^2 U(k,t) + c^2k^2 U(k,t) = 0 \quad \because D^2 = d^2/dt^2$$

The charac. equation is

$$D^2 + c^2k^2 = 0$$

$$\Rightarrow D^2 = -c^2k^2 \quad \Rightarrow D = \pm ick.$$

The general solution is

$$U(k,t) = C_1 \cos ckt + C_2 \sin ckt \quad \text{——— (3)}$$

Taking Fourier transform of (2), we get

$$\mathcal{F}\{u(x,0)\} = \mathcal{F}\{p(x)\} \text{ and } \mathcal{F}\{u_t(x,0)\} = \mathcal{F}\{q(x)\}$$

$$\Rightarrow U(k,0) = P(k) \text{ and } \frac{d}{dt} U(k,0) = Q(k). \quad \text{——— (4)}$$

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(3) $\Rightarrow U(k,0) = C_1$

(4) $\Rightarrow \boxed{C_1 = P(k)}$

Also (3)

$\Rightarrow \frac{d}{dt} U(k,t) = -C_1 \sin(ck)t \cdot ck + C_2 \cos(ck)t \cdot ck$

$\Rightarrow \frac{d}{dt} U(k,0) = ck\, C_2$

(4) $\Rightarrow ck\, C_2 = Q(k) \Rightarrow \boxed{C_2 = Q(k)/ck}$

Now equation (3) becomes

$$U(k,t) = P(k)\cos(ck)t + \frac{Q(k)}{ck}\sin(ck)t.$$

Taking inverse Fourier transform on both sides

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left\{ P(k)\cos(ck)t + \frac{Q(k)}{ck}\sin(ck)t \right\} dk.$$

Question: A/2018, S/2018.

Find the Fourier transform of $g(x) = \frac{1}{(x^2+a^2)^2}$

Proof: The function $g(x)$ as well as its derivative are continuous over the interval $(-\infty, +\infty)$, and the integral $\int_{-\infty}^{\infty} g(x)dx$ is absolutely integrable. Therefore, the Fourier transform of $g(x)$ must exists.

$$\mathcal{F}\{g(x)\} = G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \frac{1}{(x^2+a^2)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \oint_C \frac{e^{ikz}}{(z^2+a^2)^2} dz \quad \text{---- (1)}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot 2\pi i \sum_{i=1}^{2} R_i \quad \text{---- (2)}$$

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where ① has poles of order 2 at $z = \pm ai$.

Now, $R_1 = \lim_{z \to ai} \frac{d}{dz}\left[(z-ai)^2 \frac{e^{ikz}}{(z-ai)^2(z+ai)^2}\right]$

$= \lim_{z \to ai} \frac{d}{dz}\left[\frac{e^{ikz}}{(z+ai)^2}\right]$

$= \lim_{z \to ai}\left[\frac{(z+ai)^2(ik)e^{ikz} - e^{ikz}\,2(z+ai)(1)}{(z+ai)^4}\right]$

$= \lim_{z \to ai}\left[\frac{(z+ai)e^{ikz}\{(z+ai)ik - 2\}}{(z+ai)^4}\right]$

$= \lim_{z \to ai}\left[\frac{e^{ikz}\{(z+ai)ik - 2\}}{(z+ai)^3}\right]$

$= e^{-ka}\left\{\frac{(2ai)ik - 2}{(2ai)^3}\right\}.$

$= e^{-ka}\left\{\frac{-2ak - 2}{-8ai^3}\right\} = e^{-ka}\frac{(ak+1)}{4a^3i}$

Similarly, we can find

$R_2 = \lim_{z \to -ai} \frac{d}{dz}\left[(z+ai)^2 \frac{e^{ikz}}{(z-ai)^2(z+ai)^2}\right]$

$= \frac{e^{ka}(ak-1)}{4a^3i}$

Thus eq ② becomes

$G(k) = \frac{1}{\sqrt{2\pi}}\, 2\pi i\left[\frac{e^{-ka}(ak+1)}{4a^3i} + \frac{e^{ka}(ak-1)}{4a^3i}\right]$

$= \sqrt{\frac{2}{\pi}}\, 2i\left[\frac{e^{-ka}ak + e^{-ka} + e^{ka}ka - e^{ka}}{4a^3i}\right]$

$= \frac{1}{a^3}\sqrt{\frac{2}{\pi}}\left\{ak\left(\frac{e^{ka} + e^{-ka}}{2}\right) - \left(\frac{e^{ka} - e^{-ka}}{2}\right)\right\}$

$= \frac{1}{a^3}\sqrt{\frac{2}{\pi}}\left[ak\cosh(ka) - \sinh(ka)\right]$

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CHAP 8.

Green's Functions

The Dirac Delta Function. 2018/S.

It is the generalization of the Kronecker delta function and has the additional property $\sum_i a_j \delta_{ij} = a_i$.

The Dirac Delta Function is defined by the following properties.

(i) $$\delta(x-t) = \begin{cases} 0, & \text{if } x \neq t \\ \infty, & \text{if } x = t \end{cases}$$

(ii) $$\int_{-\infty}^{\infty} \delta(x-t)\,dx = 1$$

(iii) $$\int_{-\infty}^{\infty} \delta(x-t) f(x)\,dx = f(t).$$

Result:
$$\int_{x_2=0}^{x}\int_{x_1=0}^{x_2} \varphi(x_1)\,dx_1\,dx_2 = \int_{x_1=0}^{x} (x-x_1)\varphi(x_1)\,dx_1 \quad \text{---(1)}$$

Proof.

The region of integration is shown in th Fig. From the Fig we see that in the double integral on L.H.S of (1), the range of variation of the integration variable is $0 \le x_1 \le x_2$ and $0 \le x_2 \le x$.

On changing the order of integration, the range of variation of the integration variables will be $x_1 \le x_2 \le x$ and $0 \le x_1 \le x$.

Therefore,

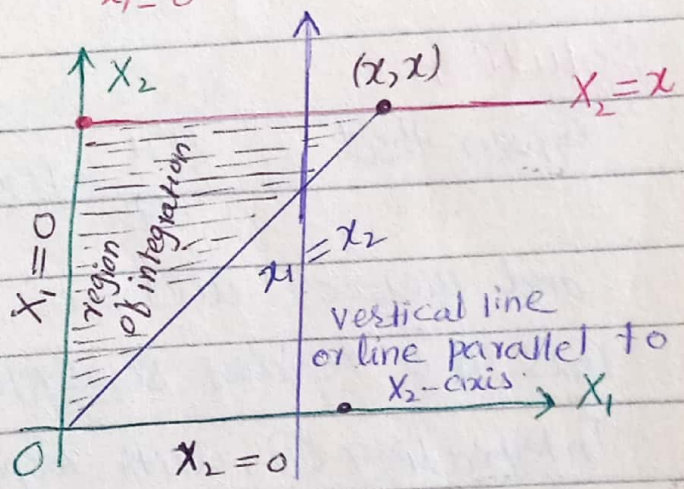

Fig.

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In Fig. minimum value of x_1 is zero and maximum value is x.

$$\int_{x_2=0}^{x}\int_{x_1=0}^{x_2} \varphi(x_1)\,dx_1\,dx_2 = \int_{x_1=0}^{x}\int_{x_2=x_1}^{x} \varphi(x_1)\,dx_2\,dx_1$$

$$= \int_{x_1=0}^{x} \varphi(x_1) \int_{x_2=x_1}^{x} dx_2\,dx_1$$

$$= \int_{x_1=0}^{x} \varphi(x_1)\,[x_2]_{x_2=x_1}^{x}\,dx_1$$

$$= \int_{x_1=0}^{x} (x-x_1)\,\varphi(x_1)\,dx_1 \quad \text{as required.}$$

Example.

Solve the problem

$$\frac{d^2u}{dx^2} = f(x)\,,\ u(0)=0\,,\ u(l)=0\,,\ 0\le x\le l.$$

Solution.

Given that $$\frac{d^2u}{dx^2} = f(x) \qquad \text{——— (1)}$$

and $u(0)=0$, $u(l)=0$, $0\le x\le l$ ——— (2)

This is a regulas SL-system with $p(x)=1$.

Integrating (1), with respect to x, from 0 to x, we get

$$\left.\frac{du}{dx}\right|_0^x = \int_0^x f(x')\,dx'$$

نوٹ:- x نہ رکھیں کیونکہ limit میں بھی x دیا ہوا ہے

$$\Rightarrow u'(x) - u'(0) = \int_0^x f(x')\,dx'.$$

Again integrating over the same limit, we get,

$$u(x)\Big|_0^x - u'(0)\,x\Big|_0^x = \int_0^x \left\{\int_0^{x^*} f(x')\,dx'\right\} dx^* \quad \text{dummy index.}$$

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given 0 , A (say)

$$\Rightarrow u(x) - u(0) - u'(0)x = \int_{x^*=0}^{x}\int_{x'=0}^{x^*} f(x')\,dx'\,dx^*$$

$$\Rightarrow u(x) - Ax = \int_{x'=0}^{x}\int_{x^*=x'}^{x} f(x')\,dx^*\,dx' \qquad \text{using last result.}$$

$$\Rightarrow u(x) - Ax = \int_{x'=0}^{x} f(x') \int_{x^*=x'}^{x} dx^*\,dx'$$

$$= \int_{x'=0}^{x} f(x') \left[x^*\right]_{x^*=x'}^{x} dx'$$

$$= \int_{x'=0}^{x} f(x')\,(x-x')\,dx'$$

$$\Rightarrow u(x) = \int_{x'=0}^{x} (x-x')\,f(x')\,dx' + Ax \qquad \text{——— (3)}$$

using B.Cs, $u(l)=0$, we get,

$$(3) \Rightarrow u(l) = \int_{x'=0}^{l} (l-x')\,f(x')\,dx' + Al$$

$$\Rightarrow 0 = \int_{x'=0}^{l} (l-x')\,f(x')\,dx' + Al.$$

$$\Rightarrow Al = -\int_{x'=0}^{l} (l-x')\,f(x')\,dx'.$$

$$\Rightarrow A = \frac{1}{l}\int_{x'=0}^{l} (x'-l)\,f(x')\,dx'.$$

Now equation (3) becomes

$$u(x) = \int_{x'=0}^{x} (x-x')\,f(x')\,dx' + \frac{x}{l}\int_{x'=0}^{l} (x'-l)\,f(x')\,dx' \qquad \text{— (4)}$$

which is the solution of the given problem.

Next we rewrite the solution (4) to introduce

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the Green's function

$$u(x) = \int_{x'=0}^{x} (x-x')f(x')dx' + \frac{x}{\ell}\int_{x'=0}^{x}(x'-\ell)f(x')dx'$$

$$+\frac{x}{\ell}\int_{x'=x}^{\ell}(x'-\ell)f(x')dx'$$

$$= \int_{x'=0}^{x}\left\{(x-x') + \frac{x}{\ell}(x'-\ell)\right\}f(x)dx' + \frac{x}{\ell}\int_{x'=x}^{\ell}(x'-\ell)f(x')dx'.$$

$$= \int_{x'=0}^{x}\left(x - x' + \frac{xx'}{\ell} - x\right)f(x')dx' + \int_{x'=x}^{\ell}\frac{x}{\ell}(x'-\ell)f(x')dx'$$

$$= \int_{x'=0}^{x}\left(\frac{xx'}{\ell} - x'\right)f(x')dx' + \int_{x'=x}^{\ell}\frac{x}{\ell}(x'-\ell)f(x')dx'$$

$$= \int_{x'=0}^{x}\left(\frac{xx' - x'\ell}{\ell}\right)f(x')dx' + \int_{x'=x}^{\ell}\frac{x}{\ell}(x'-\ell)f(x')dx'.$$

$$= \int_{x'=0}^{x}\frac{x'}{\ell}(x-\ell)f(x')dx' + \int_{x'=x}^{\ell}\frac{x}{\ell}(x'-\ell)f(x')dx'.$$

This can be written as.

$$u(x) = \int_{x'=0}^{\ell} G(x,x')f(x')dx'$$

where $$G(x,x') = \begin{cases} \frac{x'}{\ell}(x-\ell), & 0 \le x' \le x \\ \frac{x}{\ell}(x'-\ell), & x < x' \le \ell \end{cases} \quad \text{——(5)}$$

is called Green's function associated with the given problem.

Properties of Green function. 2011, 2018/S.

The Green function as defined in (5) is a function of x'. We can also write it as a

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· function of x as

$$G(x,x') = \begin{cases} \frac{x}{\ell}(x'-\ell) & , 0 \le x \le x' \\ \frac{x'}{\ell}(x-\ell) & , x' < x \le \ell. \end{cases} \quad \text{——— (6)}$$

① It is symmetric in x and x'.
i.e. $G(x,x') = G(x',x)$.

② $G(x,x')$ is a function of x satisfies the D.E. $\frac{d^2}{dx^2}G(x,x') = 0$ in each of the intervals $0 \le x \le x'$ and $x' < x \le \ell$.

③ $G(x,x')$ as a function of x satisfies the relations $G(0,x') = 0$ and $G(\ell,x') = 0$ which are same boundary conditions as those satisfied by u.

④ $G(x,x')$ is continuous function of x in the entire interval $[0,\ell]$.

⑤ The derivative $G'(x,x')$ of $G(x,x')$
i.e. $G'(x,x') = \frac{d}{dx}G(x,x')$

$$= \begin{cases} \frac{x'}{\ell}(x'-\ell) & , 0 \le x \le x' \\ \frac{x'}{\ell} & , x' < x \le \ell \end{cases}$$

as a function of x is discontinuous at $x = x'$
where $G'(x'-0,x') - G'(x'+0,x') = \frac{1}{\ell}(x'-\ell) - x'/\ell$
$= -1$ as $p(x) = 1$.

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Existence of Green's Function.

If $\lambda = 0$ is not an eigenvalue of the problem, then Green's function exists.

Example 1. 2011, (special)

Construct Green's function associated with the problem

$$u'' + \lambda u = 0, \quad u(0) = 0, \quad u(1) = 0. \qquad \text{——— (1)}$$

Solution.

(1) First we verify that $\lambda = 0$ is not an eigenvalue of the given problem.

For $\lambda = 0$, (1) $\Rightarrow u'' = 0$

$\Rightarrow u' = C_1$

$\Rightarrow u(x) = C_1 x + C_2$

$u(0) = C_2 \Rightarrow \boxed{C_2 = 0}$

$u(1) = C_1 + 0$

$\Rightarrow 0 = C_1 \Rightarrow \boxed{C_1 = 0}$

Thus $u(x) = 0$ (a trivial solution)

Hence $\lambda = 0$ is not an eigenvalue, so Green function $G(x,t)$ exists.

(2) $G(x,t)$ satisfies the DE i.e. $\frac{d^2}{dx^2} G(x,t) = 0$ in each of the sub-intervals $[0,t]$ and $[t,1]$.

Therefore, we have

$$G(x,t) = \begin{cases} C_1 x + C_2, & 0 \le x \le t \\ C_3 x + C_4, & t < x \le 1 \end{cases} \qquad \text{——— (2)}$$

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③ $G(x,t)$ satisfies the boundary conditions.

i.e. $G(0,t) = 0$, $G(1,t) = 0$.

For $G(0,t) = 0$ we have

$G(x,t) = C_1 x + C_2$

so $G(0,t) = C_2 \Rightarrow \boxed{C_2 = 0}$

For $G(1,t) = 0$ we have

$G(x,t) = C_3 x + C_4$

so $G(1,t) = C_3(1) + C_4 \Rightarrow C_3 = -C_4$

Thus ②

$$\Rightarrow G(x,t) = \begin{cases} C_1 x, & 0 \le x \le t \\ -C_4 x + C_4, & t < x \le 1 \end{cases} \quad \text{---------- ③}$$

④ $G(x,t)$ is continuous on $[0,1]$ and in particular, $G(x,t)$ is continuous at $x = t$.

So $\lim_{x \to t} G(x,t)$ exists.

$$\Rightarrow \lim_{x \to t^-} G(x,t) = \lim_{x \to t^+} G(x,t).$$

$$\Rightarrow \lim_{x \to t^-} C_1 x = \lim_{x \to t^+} (-C_4 x + C_4).$$

$$\Rightarrow C_1 t = -C_4 t + C_4$$

$$= C_4(1 - t).$$

$$\Rightarrow C_4 = \frac{C_1 t}{1 - t}$$

Thus ③ $\Rightarrow$

$$G(x,t) = \begin{cases} C_1 x, & 0 \le x \le t \\ C_1 \frac{t}{1-t}(1-x), & t < x \le 1 \end{cases} \longrightarrow ④$$

⑤ $G'(x,t)$ is discontinuous on $[0,1]$ and

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in particular, $G'(x,t)$ is discontinuous at $x=t$. For this, we have

$$G'(t-0,t) - G'(t+0,t) = +1/p(t) = +1/1 = +1$$

$$\Rightarrow G'(t-0,t) - G'(t+0,t) = +1 \quad \text{——— (5)}$$

From (4)

$$G'(x,t) = \begin{cases} C_1 & , 0 \le x \le t. \\ -C_1 \frac{t}{1-t} & , t < x \le 1. \end{cases}$$

$G'(t-0,t) - G'(t+0,t) = \frac{1}{p(t)}$ i.e. $\frac{-1}{p(t)}$

or $G'(t+0,t) - G'(t-0,t) = -\frac{1}{p(t)}$

So (5) $\Rightarrow C_1 + C_1 \frac{t}{1-t} = +1$

$$\Rightarrow \frac{C_1(1-t) + C_1(t)}{1-t} = +1$$

$$\Rightarrow C_1 - C_1 t + C_1 t = 1 - t$$

$$\Rightarrow C_1 = 1 - t$$

Thus

$$(4) \Rightarrow G(x,t) = \begin{cases} (1-t)x & , 0 \le x \le t \\ t(1-x) & , t < x \le 1. \end{cases}$$

is the required Green function.

Example 2.

Find Green's function associated with the problem.

$$xu'' + u' + \lambda x(x)u = 0, \quad \text{——— (1)}$$

$u(0)$ is finite and $u(1) = 0$.

Solution.

(1) First we verify that $\lambda = 0$ is not an

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eigenvalue of the given problem.

For $\lambda = 0$, ① $\Rightarrow x u'' + u' = 0$

$\Rightarrow \frac{d}{dx}(x u') = 0$

Integrating, we get,

$x u' = C_1 \quad \Rightarrow u' = C_1/x$

Again integrating, we get

$u(x) = C_1 \ln x + C_2$

where C_1 and C_2 are constants.

Now we apply the B.C.

$u(0)$ is finite $\Rightarrow \boxed{C_1 = 0}$

$u(1) = 0 \Rightarrow u(1) = C_1 \ln(1) + C_2 = 0$

$\Rightarrow \boxed{C_2 = 0}$

Thus $u(x) = 0$. (a trivial solution)

Hence $\lambda = 0$ is not an eigenvalue, so Green function $G(x,t)$ exists.

② $G(x,t)$ satisfies the DE

i.e. $x G'' + G' = 0$ in each of the subinterval $(0,t]$ and $(t,1]$.

Therefore, we have,

$$G(x,t) = \begin{cases} C_1 \ln x + C_2 , & 0 < x \le t \\ C_3 \ln x + C_4 , & t < x \le 1 \end{cases} \quad \text{——— (2)}$$

③ $G(x,t)$ satisfies the boundary conditions.

i.e. $G(0,t)$ is finite, and $G(1,t) = 0$.

For $G(0,t)$ is finite, we have

$G(x,t) = C_1 \ln x + C_2$

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So $G(0,t) = C_1 \ln(0) + C_2$

$\Rightarrow \boxed{C_1 = 0}$

For $G(1,t) = 0$, we have

$G(x,t) = C_3 \ln x + C_4$

$G(1,t) = C_4 \Rightarrow \boxed{C_4 = 0}$

Thus (2)

$$\Rightarrow G(x,t) = \begin{cases} C_2 & , \ 0 < x \le t \\ C_3 \ln x & , \ t < x \le 1. \end{cases} \quad \text{——— (3)}$$

(4) $G(x,t)$ is continuous on $(0,1]$ and in particular, $G(x,t)$ is continuous at $x = t$.

So $\lim_{x \to t} G(x,t)$ exists.

$\Rightarrow \lim_{x \to t^-} G(x,t) = \lim_{x \to t^+} G(x,t).$

$\Rightarrow \lim_{x \to t^-} C_2 = \lim_{x \to t^+} C_3 \ln x.$

$\Rightarrow C_2 = C_3 \ln t.$

Thus (3) $\Rightarrow$

$$G(x,t) = \begin{cases} C_3 \ln t & , \ 0 < x \le t \\ C_3 \ln x & , \ t < x \le 1. \end{cases} \quad \text{——— (4)}$$

(5) $G'(x,t)$ is discontinuous on $(0,1]$ and in particular, $G'(x,t)$ is discontinuous at $x = t$. For this, we have

$G'(t-0,t) - G'(t+0,t) = +1/p(t) = +1/t$

$\Rightarrow G'(t-0,t) - G'(t+0,t) = +1/t$ ——— (5)

From (4),

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$$G'(x,t) = \begin{cases} 0 & , 0 < x \le t \\ C_3 \cdot \frac{1}{x} & , t < x \le 1 \end{cases}$$

So (5) $\Rightarrow 0 - \frac{C_3}{x} = +\frac{1}{t} \quad \Rightarrow \frac{-C_3}{t} = +\frac{1}{t}$

$\Rightarrow \boxed{C_3 = -1}$

Thus (4) $\Rightarrow G(x,t) = \begin{cases} -\ln t & , 0 < x \le t \\ -\ln x & , t < x \le 1 \end{cases}$

is the required Green's function.

Example 3. 2012,

Construct Green's function for the B.V.P.

$$xu'' + u' - \frac{n^2}{x}u + \lambda g(x)u = 0 \quad \text{——— (1)}$$

$u(0)$ is finite and $u(1) = 0$.

Solution.

Here $p(x) = x$.

(1) First we verify that $\lambda = 0$ is not an eigenvalue of the given problem.

For $\lambda = 0$ (1) $\Rightarrow xu'' + u' - \frac{n^2}{x}u = 0$

$\Rightarrow x^2u'' + xu' - n^2u = 0.$ ——— (A)

which is the Euler's - Cauchy equation.

Subsituting $x = e^t$, we get

$\Rightarrow t = \ln x$

$\Rightarrow \frac{dt}{dx} = \frac{1}{x}$ or $\frac{dx}{dt} = x.$

So $\frac{du}{dx} = \frac{du}{dt}\cdot\frac{dt}{dx} = \frac{1}{x}\frac{du}{dt}.$

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$$\Rightarrow \frac{d}{dx} = \frac{1}{x}\frac{d}{dt} \qquad \text{(operator form)}$$

So $$\frac{d^2u}{dx^2} = \frac{d}{dx}\left(\frac{du}{dx}\right) = \frac{d}{dt}\cdot\frac{dt}{dx}\cdot\frac{dt}{dx}\cdot$$

$$= \frac{d}{dt}\left(\frac{1}{x}\frac{du}{dt}\right)\frac{1}{x}\cdot$$

$$= \frac{1}{x}\frac{d}{dt}\left(\frac{1}{x}\frac{du}{dt}\right)\cdot$$

$$= \frac{1}{x}\left[\frac{1}{x}\frac{d^2u}{dt^2} + \frac{d}{dt}\left(\frac{1}{x}\right)\frac{du}{dt}\right]$$

$$= \frac{1}{x}\left(\frac{1}{x}\frac{d^2u}{dt^2} - \frac{1}{x^2}\frac{dx}{dt}\cdot\frac{du}{dt}\right)$$

$$= \frac{1}{x}\left(\frac{1}{x}\frac{d^2u}{dt^2} - \frac{1}{x^2}\cdot x\frac{du}{dt}\right)\cdot$$

$$\Rightarrow u'' = \frac{1}{x^2}\left[\frac{d^2u}{dt^2} - \frac{du}{dt}\right]\cdot$$

Now equation (A) becomes

$$x^2\cdot\frac{1}{x^2}\left(\frac{d^2u}{dt^2} - \frac{du}{dt}\right) + x\cdot\frac{1}{x}\frac{du}{dt} - n^2u = 0\cdot$$

$$\Rightarrow \frac{d^2u}{dt^2} - \frac{du}{dt} + \frac{du}{dt} - n^2u = 0$$

$$\Rightarrow \frac{d^2u}{dt^2} - n^2u = 0\cdot$$

$$\Rightarrow D^2u - n^2u = 0 \Rightarrow D^2 - n^2 = 0$$

$$\Rightarrow D^2 = n^2 \Rightarrow D = \pm n$$

$$\Rightarrow u(t) = C_1e^{nt} + C_2e^{-nt}\cdot$$

$$\Rightarrow u(x) = C_1x^n + C_2x^{-n}$$

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Apply the B.Cs, we have,

$u(0)$ is finite $\Rightarrow$ $\boxed{C_2 = 0}$

$u(1) = 0 \Rightarrow \boxed{C_1 = 0}$

Thus $u(x) = 0$ (a trivial solution)

Hence $\lambda = 0$ is not an eigenvalue, so, Green's function $G(x,t)$ exists.

② $G(x,t)$ satisfies the D.E.

i.e. $x^2 G'' + xG' - n^2 G = 0$ in each of the subintervals, $[0, t]$ and $[t, 1]$.

Therefore, we have

$$G(x,t) = \begin{cases} C_1 x^n + C_2 x^{-n}, & 0 \le x \le t \\ C_3 x^n + C_4 x^{-n}, & 0 < x \le 1 \end{cases} \longrightarrow (2)$$

③ $G(x,t)$ satisfies the boundary conditions

i.e. $G(0,t)$ is finite, $G(1,t) = 0$.

For $G(0,t)$ is finite, we have

$C_2 = 0$

For $G(1,t) = 0$, we have

$G(1,t) = C_3 + C_4 \Rightarrow C_3 + C_4 = 0$

$\Rightarrow C_3 = -C_4$

Thus ② $\Rightarrow$

$$G(x,t) = \begin{cases} C_1 x^n, & 0 \le x \le t \\ -C_4 x^n + C_4 x^{-n}, & 0 < x \le 1 \end{cases} \longrightarrow (3)$$

④ $G(x,t)$ is continuous on $[0,1]$ and in particular, $G(x,t)$ is continuous at $x = t$.

So $\lim_{x \to t} G(x,t)$ exists.

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$$\Rightarrow \lim_{x\to t^-} C_1 x^n = \lim_{x\to t^+} C_4(x^{-n}-x^n)$$

$$\Rightarrow C_1 t^n = C_4(t^{-n}-t^n)$$

$$\Rightarrow C_1 = C_4\left(\frac{t^{-n}-t^n}{t^n}\right) = C_4\left(\frac{1-t^{2n}}{t^{2n}}\right) = C_4(t^{-2n}-1)$$

Thus (3) $\Rightarrow G(x,t) = \begin{cases} C_4(t^{-2n}-1)x^n, & 0\le x\le t \\ C_4(x^{-n}-x^n), & 0<x\le 1 \end{cases}$ ——— (4)

(5) $G'(x,t)$ is discontinuous on $[0,1]$ and in particular, $G'(x,t)$ is discontinuous at $x=t$. For this, we have

$$G'(t-0,t) - G'(t+0,t) = \frac{1}{p(t)} = 1/t.$$

$$\Rightarrow G'(t-0,t) - G'(t+0,t) = \frac{1}{t} \quad \text{——— (5)}$$

From (4)

$$G'(x,t) = \begin{cases} C_4(t^{-2n}-1)\,n\,t^{n-1}, & 0\le x\le t. \\ C_4(-n t^{-n-1} - n t^{n-1}), & 0<x\le 1. \end{cases}$$

so (5) $\Rightarrow C_4 n(t^{-2n}-1)t^{n-1} + C_4 n(t^{-n-1} + t^{n-1}) = 1/t$

$$\Rightarrow C_4 n\{t^{-n-1} - t^{n-1} + t^{-n-1} + t^{n-1}\} = \frac{1}{t}$$

$$\Rightarrow 2t^{-n-1} n C_4 = \frac{1}{t}$$

$$\Rightarrow C_4 = \frac{1}{2n t^{-n-1}\cdot t} = \frac{1}{2n t^{-n}}$$

$$\Rightarrow \boxed{C_4 = \frac{t^n}{2n}}$$

Thus (4) $\Rightarrow G(x,t) = \begin{cases} t^n/_{2n}\,(t^{-2n}-1)x^n, & 0\le x\le t \\ t^n/_{2n}\,(x^{-n}-x^n), & 0<x\le 1. \end{cases}$

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$$\Rightarrow G(x,t) = \begin{cases} \frac{1}{2n}(t^{-n} - t^{n})x^{n}, & 0 \le x \le t \\ \frac{1}{2n}(x^{-n} - x^{n})t^{n}, & 0 < x \le 1. \end{cases}$$

which is the required Green's function.

Example 4. 2013, 2014, 2016,

Construct Green's function for the B.V.P.

$$\frac{d}{dx}\{(1-x^2)u'\} - \frac{h^2}{1-x^2}u + \lambda r(x)u = 0 \quad \text{---(1)}$$

$u(\pm 1)$ are finite.

Solution.

Here $p(x) = 1-x^2$.

(1) First we verify that $\lambda = 0$ is not an eigenvalue of the given problem.

For $\lambda = 0$, (1) $\Rightarrow \frac{d}{dx}\{(1-x^2)u'\} - \frac{h^2}{1-x^2}u = 0.$

$$\Rightarrow (1-x^2)u'' + (-2x)u' - \frac{h^2}{1-x^2}u = 0.$$

$$\Rightarrow (1-x^2)u'' - 2xu' - \frac{h^2}{1-x^2}u = 0 \quad \text{---(A)}$$

which is the Cauchy-Euler equation.

Subsituting $\frac{1+x}{1-x} = e^{t}$, we get.

$$\Rightarrow t = \ln\left(\frac{1+x}{1-x}\right)$$

$$= \ln(1+x) - \ln(1-x)$$

$$\Rightarrow \frac{dt}{dx} = \frac{1}{1+x} - \frac{1}{1-x}(-1)$$

$$= \frac{1}{1+x} + \frac{1}{1-x} = \frac{1-x+1+x}{1-x^2}$$

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$$\Rightarrow \frac{dt}{dx} = \frac{2}{1-x^2}.$$

so $\frac{du}{dx} = \frac{du}{dt}\cdot\frac{dt}{dx} = \frac{2}{1-x^2}\frac{du}{dt}$.

$$\Rightarrow \frac{du}{dx} = \frac{2}{1-x^2}\frac{du}{dt}.$$

$$\Rightarrow \frac{d}{dx} = \frac{2}{1-x^2}\frac{d}{dt} \quad \text{(operator form)}$$

so $\frac{d^2u}{dx^2} = \frac{d}{dx}\left(\frac{du}{dx}\right) = \frac{d}{dt}\left(\frac{du}{dx}\right)\frac{dt}{dx}$.

$$= \frac{d}{dt}\left(\frac{2}{1-x^2}\frac{du}{dt}\right)\frac{2}{1-x^2}.$$

$$= \frac{2}{1-x^2}\left\{\frac{2}{1-x^2}\frac{d^2u}{dt^2} + \frac{(-2)}{(1-x^2)^2}\cdot -2x\cdot\frac{dx}{dt}\cdot\frac{du}{dt}\right\}.$$

$$= \frac{2}{1-x^2}\left\{\frac{2}{1-x^2}\frac{d^2u}{dt^2} + \frac{4x}{(1-x^2)^2}\cdot\left(\frac{1-x^2}{2}\right)\frac{du}{dt}\right\}.$$

$$\Rightarrow u'' = \frac{4}{(1-x^2)^2}\left\{\frac{d^2u}{dt^2} + x\frac{du}{dt}\right\}$$

Now equation (A) becomes

$$(1-x^2)\cdot\frac{4}{(1-x^2)^2}\left\{\frac{d^2u}{dt^2} + x\frac{du}{dt}\right\} - 2x\cdot\frac{2}{1-x^2}\frac{du}{dt}$$

$$-\frac{h^2}{1-x^2}u = 0$$

$$4\frac{d^2u}{dt^2} + 4x\frac{du}{dt} - 4x\frac{du}{dt} - h^2u = 0.$$

$$\Rightarrow 4\frac{d^2u}{dt^2} - h^2u = 0.$$

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$$\Rightarrow \frac{d^2u}{dt^2} - \frac{h^2}{4}u = 0.$$

$$\Rightarrow D^2 - \frac{h^2}{4} = 0 \quad \Rightarrow D^2 = h^2/4$$

$$\Rightarrow D = \pm h/2$$

$$\Rightarrow u(t) = C_1 e^{\frac{h}{2}t} + C_2 e^{-\frac{h}{2}t}$$

$$= C_1 (e^t)^{h/2} + C_2 (e^t)^{-\frac{h}{2}}$$

By back substitution,

$$u(x) = C_1 \left(\frac{1+x}{1-x}\right)^{\frac{h}{2}} + C_2 \left(\frac{1+x}{1-x}\right)^{-h/2}.$$

Applying B.Cs, we get

$u(1)$ is finite $\Rightarrow \boxed{C_1 = 0}$

$u(-1)$ is finite $\Rightarrow \boxed{C_2 = 0}$

Thus $u(x) = 0$ (a trivial solution).

Hence $\lambda = 0$ is not an eigenvalue, so Green function $G(x,t)$ exists.

(2) $G(x,t)$ satisfies the D.E.

i.e. $(1-x^2)G'' - 2xG' - \frac{h^2}{1-x^2}G = 0$ in each of the sub-interval $[-1, t]$ and $[t, 1]$.

Therefore, we have

$$G(x,t) = \begin{cases} C_1 \left(\frac{1+x}{1-x}\right)^{h/2} + C_2 \left(\frac{1+x}{1-x}\right)^{-\frac{h}{2}}, & -1 \le x \le t \\ C_3 \left(\frac{1+x}{1-x}\right)^{h/2} + C_4 \left(\frac{1+x}{1-x}\right)^{-h/2}, & t < x \le 1. \end{cases} \longrightarrow (2)$$

(3) $G(x,t)$ satisfies the boundary conditions.

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i.e. $G(\pm 1, t)$ are finite.

For $G(-1, t)$ is finite, we have

$$\boxed{C_2 = 0}$$

For $G(1, t)$ is finite, we have

$$\boxed{C_3 = 0}$$

Thus ② $\Rightarrow G(x,t) = \begin{cases} C_1\left(\frac{1+x}{1-x}\right)^{h/2}, & -1 \le x \le t \\ C_4\left(\frac{1+x}{1-x}\right)^{-h/2}, & t < x \le 1. \end{cases}$ ——→ ③

④ $G(x,t)$ is continuous on $[-1, 1]$ and in particular, $G(x,t)$ is continuous at $x = t$.

So $\lim_{x \to t} G(x,t)$ exists.

$$\Rightarrow \lim_{x \to t^-} G(x,t) = \lim_{x \to t^+} G(x,t).$$

$$\Rightarrow \lim_{x \to t^-} C_1\left(\frac{1+x}{1-x}\right)^{h/2} = \lim_{x \to t^+} C_4\left(\frac{1+x}{1+x}\right)^{-h/2}$$

$$\Rightarrow C_1\left(\frac{1+t}{1-t}\right)^{h/2} = C_4\left(\frac{1+t}{1-t}\right)^{-h/2}.$$

$$\Rightarrow C_1\left(\frac{1+t}{1-t}\right)^{h/2} = C_4\left(\frac{1-t}{1+t}\right)^{h/2}$$

$$\Rightarrow C_1 = C_4\left(\frac{1-t}{1+t}\right)^{h/2}\left(\frac{1+t}{1-t}\right)^{-h/2}.$$

$$\Rightarrow C_1 = C_4\left(\frac{1-t}{1+t}\right)^{h}.$$

Thus ③ $\Rightarrow G(x,t) = \begin{cases} C_4\left(\frac{1-t}{1+t}\right)^{h}\left(\frac{1+x}{1-x}\right)^{+h/2}, & -1 \le x \le t \\ C_4\left(\frac{1+x}{1-x}\right)^{-h/2}, & t < x \le 1. \end{cases}$ ——→ ④

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(5) $G'(x,t)$ is discontinuous on $[-1,1]$ and in particular, $G'(x,t)$ is discontinuous at $x=t$.

For this, we have

$$G'(t-0,t) - G'(t+0,t) = 1/p(t)$$

$$\Rightarrow G'(t-0,t) - G'(t+0,t) = \frac{1}{1-t^2} \quad \text{---(5)}$$

From (4)

$$G'(x,t) = \begin{cases} C_4 h \left(\frac{1+t}{1-t}\right)^{\frac{h}{2}-1} \cdot \frac{1}{(1-t)^2} \cdot \left(\frac{1-t}{1+t}\right)^h, & -1 \le x \le t \\ -C_4 h \left(\frac{1+t}{1-t}\right)^{-\frac{h}{2}-1} \cdot \frac{1}{(1-t)^2}, & t \le x \le 1. \end{cases}$$

So (5) $\Rightarrow C_4 h \left(\frac{1+t}{1-t}\right)^{\frac{h}{2}} \cdot \left(\frac{1+t}{1-t}\right)^{-1} \cdot \frac{1}{(1-t)^2} \left(\frac{1+t}{1-t}\right)^{-h}$

$$+ C_4 h \left(\frac{1+t}{1-t}\right)^{\frac{-h}{2}} \cdot \left(\frac{1+t}{1-t}\right)^{-1} \cdot \frac{1}{(1-t)^2} = \frac{1}{1-t^2}$$

$$\Rightarrow C_4 h \left\{ \left(\frac{1+t}{1-t}\right)^{-h/2} \left(\frac{1-t}{1+t}\right) \cdot \frac{1}{(1-t)^2} \right.$$

$$\left. + \left(\frac{1+t}{1-t}\right)^{-\frac{h}{2}} \left(\frac{1-t}{1+t}\right) \cdot \frac{1}{(1-t)^2} \right\} = \frac{1}{1-t^2}$$

$$\Rightarrow 2 C_4 h \left(\frac{1-t}{1+t}\right)^{\frac{h}{2}+1} = \frac{1}{(1-t)(1+t)} \cdot (1-t)^2$$

$$\Rightarrow 2 C_4 h \left(\frac{1-t}{1+t}\right)^{\frac{h}{2}} \left(\frac{1-t}{1+t}\right) = \frac{1-t}{1+t}$$

$$\Rightarrow 2 C_4 h \left(\frac{1-t}{1+t}\right)^{\frac{h}{2}} = 1$$

$$\Rightarrow C_4 = \frac{1}{2h} \left(\frac{1+t}{1-t}\right)^{h/2}.$$

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Thus (4) $\Rightarrow$

$$G(x,t) = \begin{cases} \frac{1}{2h}\left(\frac{1+t}{1-t}\right)^{h/2}\left(\frac{1-t}{1+t}\right)^{h}\left(\frac{1+x}{1-x}\right)^{h/2}, & -1 \le x \le t. \\ \frac{1}{2h}\left(\frac{1+t}{1-t}\right)^{h/2}\left(\frac{1-x}{1+x}\right)^{h/2}, & t < x \le 1. \end{cases}$$

$$= \begin{cases} \frac{1}{2h}\left(\frac{1-t}{1+t}\right)^{h/2}\left(\frac{1+x}{1-x}\right)^{h/2}, & -1 \le x \le t. \\ \frac{1}{2h}\left(\frac{1+t}{1-t}\right)^{h/2}\left(\frac{1-x}{1+x}\right)^{h/2}, & t < x \le 1. \end{cases}$$

which is the required Green function.

Example 5. Special

Find Green function for the B.V.P.

$u'' + \lambda u = 0$, $u(0) + u'(1) = 0$, $u(1) + 2u'(0) = 0$. ——— (1)

Solution.

Here $p(x) = 1$.

(1) First we verify that $\lambda = 0$ is not an eigenvalue of the given problem.

For $\lambda = 0$ (1) $\Rightarrow u'' = 0$

$\Rightarrow u' = C_1$

$\Rightarrow u(x) = C_1 x + C_2$

Applying the B.Cs, we get.

$u(0) + u'(1) = 0$

$\Rightarrow C_2 + C_1 = 0 \quad \Rightarrow C_2 = -C_1$

and $u(1) + 2u'(0) = 0$

$\Rightarrow C_1 + C_2 + 2C_1 = 0$

$\Rightarrow 3C_1 + C_2 = 0$

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$\Rightarrow 3C_1 - C_1 = 0 \quad \because C_2 = -C_1$

$\Rightarrow 2C_1 = 0 \Rightarrow \boxed{C_1 = 0}$

$\Rightarrow \boxed{C_2 = 0}$

Thus $u(x) = 0$ (a trivial solution)

Hence $\lambda = 0$ is not an eigenvalue, so Green's function $G(x,t)$ exists.

② $G(x,t)$ satisfies the DE.

i.e. $G'' = 0$ in each of the sub-intervals $[0,t]$ and $(t,1]$. Therefore, we have

$$G(x,t) = \begin{cases} C_1 x + C_2, & 0 \le x \le t \\ C_3 x + C_4, & t \le x \le 1 \end{cases} \quad \text{——— (2)}$$

③ $G(x,t)$ satisfies the boundary conditions.

i.e. $G(0,t) + G'(1,t) = 0$

and $G(1,t) + 2G'(0,t) = 0$

For $G(0,t) + G'(1,t) = 0$, we have

$C_2 + C_3 = 0$ —— (i) $\because G(0,t) = C_2$, $G'(1,t) = C_3$

For $G(1,t) + 2G'(0,t) = 0$, we have

$C_3 + C_4 + 2C_1 = 0$ —— (ii)

(i) $\Rightarrow C_2 = -C_3$

(ii) $\Rightarrow -C_2 + C_4 + 2C_1 = 0$

$\Rightarrow C_4 = C_2 - 2C_1$

Thus

$$② \Rightarrow G(x,t) = \begin{cases} C_1 x + C_2, & 0 \le x \le t \\ -C_2 x + C_2 - 2C_1, & t \le x \le 1 \end{cases} \quad \text{——— (3)}$$

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④ $G(x,t)$ is continuous on $[0,1]$ and in particular, $G(x,t)$ is continuous at $x=t$.

So $\lim_{x\to t} G(x,t)$ exists.

$$\lim_{x\to t^-} G(x,t) = \lim_{x\to t^+} G(x,t)$$

$$\Rightarrow \lim_{x\to t^-} (C_1 x + C_2) = \lim_{x\to t^+} (C_2 - C_2 x - 2C_1)$$

$$\Rightarrow C_1 t + C_2 = C_2 - C_2 t - 2C_1$$

$$\Rightarrow C_1 t = -C_2 t - 2C_1$$

$$\Rightarrow C_1 t + 2C_1 = -C_2 t$$

$$\Rightarrow C_1 (t+2) = -C_2 t$$

$$\Rightarrow C_1 = \frac{-t}{t+2} C_2.$$

Thus ③ $\Rightarrow G(x,t) = \begin{cases} \frac{-t}{t+2} x C_2 + C_2, & 0 \le x \le t \\ C_2(1-x) + \frac{2t}{t+2} C_2, & t < x \le 1. \end{cases}$ ——— ④

⑤ $G'(x,t)$ is discontinuous on $[0,1]$ and in particular, $G'(x,t)$ is discontinuous at $x=t$.

For this, we have

$$G'(t-0,t) - G'(t+0,t) = 1/p(t) = 1/1 = 1$$

$$\Rightarrow G'(t-0,t) - G'(t+0,t) = 1 \quad \text{——— ⑤}$$

From ④ $G'(x,t) = \begin{cases} \frac{-t}{t+2} C_2, & 0 \le x \le t \\ C_2(-1), & t < x \le 1 \end{cases}$

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so (5) $\Rightarrow -\frac{t}{t+2}C_2 + C_2 = 1$.

$\Rightarrow C_2\left\{\frac{-t}{t+2}+1\right\}=1 \Rightarrow C_2\left\{\frac{-t+t+2}{t+2}\right\}=1$.

$\Rightarrow C_2 = \frac{t+2}{2}$.

Thus (4) $\Rightarrow G(x,t) = \begin{cases} \left(-\frac{t}{t+2}x+1\right)\frac{t+2}{2}, & 0\le x\le t. \\ \left(1-x+\frac{2t}{t+2}\right)\left(\frac{t+2}{2}\right), & t\le x\le 1. \end{cases}$

$= \begin{cases} -\frac{1}{2}(tx-t-2), & 0\le x\le t \\ -\frac{1}{2}(tx+2x-3t-2), & t\le x\le 1. \end{cases}$

which is required Green's function.

Modified Green's function.

When $\lambda=0$ is an eigenvalue of the SL-system defined by

$L(u)+\lambda s(x)u=0$, $B_1(u)=0$, $B_2(u)=0$.

Then the associated Green's function, called modified Green's function.

It will be denoted by $G_M(x,t)$ and can be constructed as 2012, 2015 (write steps).

(1) Let $u_0(x)$ be the normalized eigenfunction corresponding to $\lambda=0$.

This means that

$$\langle u_0, u_0\rangle = \int_a^b u_0(x)u_0(x)\,dx = 1$$

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② The modified Green function $G_M(x,t)$ satisfies the equation

$L[G_M(x,t)] = u_0(x)u_0(t)$. in each of the sub-intervals $[a,t]$ and $(t,b]$.

③ $G_M(x,t)$ satisfies the B.Cs.

$B_1[G_M(x,t)] = 0$, $B_2[G_M(x,t)] = 0$.

④ $G_M(x,t)$ is continuous everywhere in the interval $[a,b]$, and in particular at $x=t$.

⑤ The modified Green function satisfies the discontinuity condition

$G'_M(t-0,t) - G'_M(t+0,t) = \frac{1}{p(t)}$.

⑥ The modified Green function satisfies the orthogonality condition

$\int_a^b G_M(x,t)u_0(x)dx = 0$.

Example 1.

Construct Green's function associated with the system

$u'' + \lambda s(x)u = 0$, $u'(0) = 0 = u'(1)$. —①

Solution.

First we will check that $\lambda = 0$ is an eigenvalue or not?

For $\lambda = 0$ ① $\Rightarrow u''(x) = 0$

$\Rightarrow u' = C_1$

$\Rightarrow u(x) = C_1x + C_2$.

Applying the boundary conditions, we get,

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$u'(0) = 0 \Rightarrow \boxed{C_1 = 0}$

$u'(1) = 0 \Rightarrow \boxed{C_1 = 0}$

$\Rightarrow u(x) = C_2$ (a non trivial solution)

So $\lambda = 0$ is an eigenvalue of this problem.

Thus, the modified Green's function $G_M(x,t)$ exists.

(1) First we have to normalized our eigenfunction $u_0(x)$ corresponding to $\lambda = 0$.

$\Rightarrow \int_0^1 u_0(x)\, u_0(x)\, dx = 1$

$\Rightarrow \int_0^1 C_2 C_2\, dx = 1 \quad = \int_0^1 C_2^2\, dx = 1.$

$\Rightarrow C_2^2 x\Big|_0^1 = 1 \quad \Rightarrow C_2^2(1) = 1 \quad \Rightarrow C_2^2 = 1$

$\Rightarrow C_2 = 1.$ (take only +ve value)

$\Rightarrow u_0(x) = 1.$

(2) $G_M(x,t)$ satisfies the D.E.

$G_M''(x,t) = u_0(x)\, u_0(t)$

$= (1)(1) = 1$

$\Rightarrow G_M''(x,t) = 1$ in each of the sub-intervals $[0,t]$ and $(t,1]$

On integration,

$$G_M'(x,t) = \begin{cases} x + C_1 & , 0 \le x \le t \\ x + C_2 & , t < x \le 1. \end{cases} \longrightarrow (2)$$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + C_1 x + C_3 & , 0 \le x \le t \\ \frac{x^2}{2} + C_2 x + C_4 & , t < x \le 1. \end{cases} \longrightarrow (3)$$

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③ $G_M(x,t)$ satisfies the B.C's.

i.e. $G_M'(0,t) = 0 = G_M'(1,t)$.

From ②

$\Rightarrow G'(0,t) = 0 \Rightarrow \boxed{C_1 = 0}$

$G'(1,t) = 0 \Rightarrow C_2 + 1 = 0 \Rightarrow \boxed{C_2 = -1}$

Thus ③

$$\Rightarrow G(x,t) = \begin{cases} \frac{x^2}{2} + C_3, & 0 \le x \le t \\ \frac{x^2}{2} - x + C_4, & t < x \le 1. \end{cases} \quad \text{---- ④}$$

④ $G_M(x,t)$ is continuous on $[0,1]$ and in particular, $G_M(x,t)$ is continuous at $x=t$.

So $\lim_{x \to t} G_M(x,t)$ exists.

$$\lim_{x \to t^-} G_M(x,t) = \lim_{x \to t^+} G_M(x,t).$$

$$\Rightarrow \lim_{x \to t^-} \left(\frac{x^2}{2} + C_3\right) = \lim_{x \to t^+} \left(\frac{x^2}{2} - x + C_4\right)$$

$$\Rightarrow \frac{t^2}{2} + C_3 = \frac{t^2}{2} - t + C_4$$

$$\Rightarrow C_3 = -t + C_4.$$

Thus ④

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} - t + C_4, & 0 \le x \le t \\ \frac{x^2}{2} - x + C_4, & t < x \le 1. \end{cases} \quad \text{---- ⑤}$$

⑤ The discontinuity condition does not help in determining the unknown constants.

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⑥ $G_M(x,t)$ satisfies the orthogonality condition.

i.e. $\int_0^1 G_M(x,t)\,u_0(x)\,dx = 0$.

$$\Rightarrow \int_0^t G_M(x,t)\,(1)\,dx + \int_t^1 G_M(x,t)\,(1)\,dx = 0.$$

$$\Rightarrow \int_0^t \left(\frac{x^2}{2} - t + C_4\right)dx + \int_t^1 \left(\frac{x^2}{2} - x + C_4\right)dx = 0.$$

$$\Rightarrow \frac{x^3}{6} - tx + C_4 x \Big|_0^t + \frac{x^3}{6} - \frac{x^2}{2} + C_4 x \Big|_t^1 = 0.$$

$$\Rightarrow \frac{t^3}{6} - t^2 + C_4 t + \frac{1}{6} - \frac{1}{2} + C_4 - \left(\frac{t^3}{6} - \frac{t^2}{2} + C_4 t\right) = 0.$$

$$\Rightarrow \frac{t^3}{6} - t^2 + C_4 t + \frac{1}{6} - \frac{1}{2} + C_4 - \frac{t^3}{6} + \frac{t^2}{2} - C_4 t = 0$$

$$\Rightarrow -\frac{t^2}{2} - \frac{1}{3} + C_4 = 0 \Rightarrow C_4 = \frac{t^2}{2} + \frac{1}{3}.$$

Thus ⑤

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} - t + \frac{t^2}{2} + \frac{1}{3}, & 0 \le x \le t \\ \frac{x^2}{2} - x + \frac{t^2}{2} + \frac{1}{3}, & t < x \le 1 \end{cases}$$

which is required modified Green's function.

Example 2 (Special) 2018/S

Find Green's function associated with the problem

$$u'' + \lambda u = 0,\ u(0) = u(1),\ u'(0) = u'(1). \quad \text{①}$$

Solution.

First we will check that $\lambda = 0$ is an eigenvalue or not?

For $\lambda = 0$ ① $\Rightarrow u'' = 0 \Rightarrow u = C_1 x + C_2$

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Applying the B.Cs, we get

$u(0) = u(1) \Rightarrow C_2 = C_1 + C_2 \Rightarrow \boxed{C_1 = 0}$

$u'(0) = u'(1) \Rightarrow C_1 = C_1 \Rightarrow \boxed{C_1 = 0}$

$\Rightarrow u(x) = C_2$ (a non trivial solution)

So $\lambda = 0$ is an eigenvalue of this problem.

Thus, the corresponding Green's function is modified Green's function.

① First we have to normalized our eigenfunction $u_0(x)$ corresponding to $\lambda = 0$.

$$\Rightarrow \int_0^1 u_0(x)\, u_0(x)\, dx = 1$$

$$\int_0^1 C_2^2\, dx = 1 \Rightarrow C_2 = 1.$$

$\Rightarrow u_0(x) = 1.$

② $G_M(x,t)$ satisfies the D.E.

$G_M''(x,t) = u_0(x)\, u_0(t) = 1.$

$\Rightarrow G_M''(x,t) = 1$ in each of sub-intervals $[0,t]$ and $(t,1]$.

On integration

$$G_M'(x,t) = \begin{cases} x + C_1, & 0 \le x \le t \\ x + C_2, & t < x \le 1. \end{cases} \quad \text{---(2)}$$

$$\Rightarrow G(x,t) = \begin{cases} \frac{x^2}{2} + C_1 x + C_3 \\ \frac{x^2}{2} + C_2 x + C_4 \end{cases} \quad \text{---(3)}$$

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③ $G_M(x,t)$ satisfies the B.C's.

i.e. $G_M(0,t) = G_M(1,t)$

and $G'_M(0,t) = G'_M(1,t)$

so ②

$\Rightarrow C_3 = \frac{1}{2} + C_2 + C_4$

and $C_4 = 1 + C_2$

Thus ③ $\Rightarrow G_1(x,t) = \begin{cases} \frac{x^2}{2} + (1+C_2)x + \frac{1}{2} + C_2 + C_4, & 0 \le x \le t \\ \frac{x^2}{2} + C_2 x + C_4, & t < x \le 1 \end{cases}$ ——— ④

④ $G_M(x,t)$ is continuous on $[0,1]$ and in particular, $G_M(x,t)$ is continuous at $x=t$.

So $\lim_{x \to t} G_M(x,t)$ exists.

$\Rightarrow \lim_{x \to t^-} \left(\frac{x^2}{2} + (C_2+1)x + \frac{1}{2} + C_2 + C_4 \right) dx$

$= \lim_{x \to t^+} \left(\frac{x^2}{2} + C_2 x + C_4 \right)$.

$\Rightarrow \frac{t^2}{2} + (C_2+1)t + \frac{1}{2} + C_2 + C_4 = \frac{t^2}{2} + C_2 t + C_4$

$\Rightarrow C_2 t + t + \frac{1}{2} + C_2 = C_2 t$

$\Rightarrow t + \frac{1}{2} + C_2$ $\qquad C_2 + 1 = -t + \frac{1}{2}$

$\Rightarrow C_2 = -t - \frac{1}{2}$.

Thus ④ $\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + (\frac{1}{2} - t)x + \frac{1}{2} - t - \frac{1}{2} + C_4, & 0 \le x \le t \\ \frac{x^2}{2} + (-t - \frac{1}{2})x + C_4, & t < x \le 1 \end{cases}$

$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + (\frac{1}{2} - t)x - t + C_4, & 0 \le x \le t \\ \frac{x^2}{2} - (\frac{1}{2} + t)x + C_4, & t \le x \le 1 \end{cases}$ ——— ⑤

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(5) The discontinuity condition does not help in determining the unknown constant

(6) $G_m(x,t)$ satisfies the orthogonality condition

i.e. $\int_0^1 G_m(x,t)\,u_0(x)\,dx = 0$

$$\Rightarrow \int_0^t G_m''(x,t)(1)dx + \int_t^1 G_m(x,t)(1)dx = 0.$$

$$\Rightarrow \int_0^t \left\{\frac{x^2}{2} + \left(\frac{1}{2}-t\right)x - t + C_4\right\}dx + \int_t^1 \left\{\frac{x^2}{2} - \left(\frac{1}{2}+t\right)x + C_4\right\}dx = 0$$

$$\Rightarrow \frac{x^3}{6} + \left(\frac{1}{2}-t\right)\frac{x^2}{2} - tx + C_4 x\Big|_0^t + \frac{x^3}{6} - \left(t+\frac{1}{2}\right)\frac{x^2}{2} + C_4 x\Big|_t^1 = 0$$

$$\Rightarrow \frac{t^3}{6} + \left(\frac{1}{2}-t\right)\frac{t^2}{2} - t^2 + C_4 t + \frac{1}{6} - \left(t+\frac{1}{2}\right)\left(\frac{1}{2}\right) + C_4 - \left(\frac{t^3}{6} - \left(t+\frac{1}{2}\right)\frac{t^2}{2} + C_4 t\right) = 0$$

$$\Rightarrow \frac{t^3}{6} + \left(\frac{t^2}{4} - \frac{t^3}{2}\right) - t^2 + C_4 t + \frac{1}{6} - \frac{t}{2} - \frac{1}{4} + C_4 - \frac{t^3}{6} + \frac{t^3}{2} + \frac{t^2}{4} - C_4 t = 0$$

$$\Rightarrow -\frac{t^2}{2} - \frac{t}{2} - \frac{1}{12} + C_4 = 0$$

$$\Rightarrow C_4 = \frac{t^2}{2} + \frac{t}{2} + \frac{1}{12}$$

Thus

(5) $\Rightarrow G_m(x,t) = \begin{cases} \frac{x^2}{2} + \left(\frac{1}{2}-t\right)x - t + \frac{t^2}{2} + \frac{t}{2} + \frac{1}{12}, & 0 \le x \le t \\ \frac{x^2}{2} - \left(\frac{1}{2}+t\right)x + \frac{t^2}{2} + \frac{t}{2} + \frac{1}{12}, & t < x \le 1. \end{cases}$

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Example 3.

Find Green's function for the following problem

$u'' + \lambda u = 0, \quad u(-1) = u(1), \quad u'(-1) = u'(1)$ ——— (1)

Solution.

First will check that $\lambda = 0$ is an eigenvalue of the given problem or not?

For $\lambda = 0$ (1) $\Rightarrow u'' = 0 \Rightarrow u(x) = C_1 x + C_2$.

Applying B.Cs, we get,

$u(-1) = u(1) \Rightarrow -C_1 + C_2 = C_1 + C_2$

$\Rightarrow 2C_1 = 0 \Rightarrow \boxed{C_1 = 0}$

$u'(-1) = u'(1) \Rightarrow C_1 = C_1 \Rightarrow \boxed{C_1 = 0}$

$\Rightarrow u(x) = C_2$. (a non trivial solution)

So $\lambda = 0$ is an eigenvalue of this problem.

Thus, the corresponding Green's function is modified Green's function.

(1) First we have to normalized our eigenfunction $u_0(x)$ corresponding to $\lambda = 0$.

$$\Rightarrow \int_{-1}^{1} u_0(x)\, u_0(x)\, dx = 1.$$

$$\Rightarrow C_2^2 x \Big|_{-1}^{1} = 1 \Rightarrow C_2^2 (2) = 1$$

$$\Rightarrow C_2^2 = 1/2 \Rightarrow C_2 = 1/\sqrt{2}$$

$$\Rightarrow u_0(x) = \frac{1}{\sqrt{2}}.$$

(2) $G_m(x,t)$ satisfies the D.E.

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$G''_m(x,t) = u_0(x)\,u_0(t) = 1/2$ in each of the sub-intervals $[-1,t]$, $(t,1]$.

On integration, we get

$$G'(x,t) = \begin{cases} \frac{1}{2}x + C_1 & , -1 \le x \le t \\ \frac{1}{2}x + C_2 & t < x \le 1 \end{cases} \quad \text{——— (2)}$$

$$\Rightarrow G(x,t) = \begin{cases} \frac{x^2}{4} + C_1 x + C_3 & , -1 \le x \le t \\ \frac{x^2}{4} + C_2 x + C_4 & , t < x \le 1 \end{cases} \quad \text{——— (3)}$$

(3) $G_m(x,t)$ satisfies the B.C.

i.e. $G(-1,t) = G(1,t)$

$$\frac{1}{4} - C_1 + C_3 = \frac{1}{4} + C_2 + C_4$$

$$\Rightarrow C_3 - C_1 = C_2 + C_4 \quad \text{——— (*)}$$

and $G'(-1,t) = G'(1,t)$.

$$\Rightarrow -\frac{1}{2} + C_1 = \frac{1}{2} + C_2$$

$$\Rightarrow C_1 = 1 + C_2 \quad \text{——— (**)}$$

(**) in (*) $\Rightarrow C_3 = 1 + C_2 + C_2 + C_4$

$$\Rightarrow C_3 = C_4 + 2C_2 + 1.$$

Thus

$$(3) \Rightarrow G_m(x,t) = \begin{cases} \frac{x^2}{4} + (C_2+1)x + C_4 + 2C_2 + 1, & -1 \le x \le t \\ \frac{x^2}{4} + C_2 x + C_4, & t < x \le 1 \end{cases} \quad \text{——— (4)}$$

(4) $G_m(x,t)$ is continuous on $[-1,1]$ and in particular at $x = t$.

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So $\lim_{x \to t} G_m(x,t)$ exists.

$\Rightarrow \lim_{x \to t^-} G_m(x,t) = \lim_{x \to t^+} G_m(x,t)$

$\Rightarrow \lim_{x \to t^-} \left\{ \frac{x^2}{4} + (C_2+1)x + C_4 + 2C_2 + 1 \right\} = \lim_{x \to t^+} \left\{ \frac{x^2}{4} + C_2 x + C_4 \right\}$

$\Rightarrow \frac{t^2}{4} + (C_2+1)t + C_4 + 2C_2 + 1 = \frac{t^2}{4} + C_2 t + C_4$

$\Rightarrow C_2 t + t + C_4 + 2C_2 + 1 = C_2 t + C_4$

$\Rightarrow t + 2C_2 + 1 = 0.$

$\Rightarrow 2C_2 = -t - 1 \qquad \because C_2 + 1 = -\frac{t}{2} - \frac{1}{2} + 1$

$\Rightarrow C_2 = -\frac{1}{2}(t+1).$

Thus (4) $\Rightarrow G_m(x,t) = \begin{cases} \frac{x^2}{4} + \left(\frac{1}{2} - \frac{t}{2}\right)x + C_4 + (-t-1) + 1, & -1 \le x \le t \\ \frac{x^2}{4} - \frac{1}{2}(t+1)x + C_4, & t < x \le 1. \end{cases}$

$\Rightarrow G_m(x,t) = \begin{cases} \frac{x^2}{4} + \frac{1}{2}(1-t)x - t + C_4, & -1 \le x \le t \\ \frac{x^2}{4} - \frac{1}{2}(1+t)x + C_4, & t \le x \le 1. \end{cases}$ ——— (5)

(5) The discontinuity condition does not help in determining the unknown constant.

(6) $G_m(x,t)$ satisfies the orthogonality condition.

i.e. $\int_{-1}^{1} G_m(x,t)\, u_0(x)\, dx = 0.$

$\Rightarrow \int_{-1}^{t} G_m(x,t) \left(\frac{1}{\sqrt{2}}\right) dx + \int_{t}^{1} G_m(x,t) \left(\frac{1}{\sqrt{2}}\right) dx = 0$

$\Rightarrow \frac{1}{\sqrt{2}} \int_{-1}^{t_2} \left(\frac{x^2}{4} + \frac{1}{2}(1-t)x - t + C_4\right) dx + \frac{1}{2} \int_{t}^{1} \left(\frac{x^2}{4} - \frac{1}{2}(1+t)x + C_4\right) dx = 0.$

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$$\Rightarrow \frac{x^3}{12} + \frac{1}{2}(1-t)\frac{x^2}{2} - tx + C_4 x \Big|_{-1}^{t} + \frac{x}{12} - \frac{1}{2}(1+t)\frac{x^2}{2} + C_4 x \Big|_{t}^{1} = 0$$

$$\Rightarrow \frac{t^3}{12} + \frac{1}{4}(1-t)t^2 - t^2 + C_4 t - \left(-\frac{1}{12} + \frac{1}{4}(1-t) + t - C_4\right)$$

$$+ \frac{1}{12} - \frac{1}{4}(1+t) + C_4 - \left(\frac{t^3}{12} - \frac{1}{4}(1+t)t^2 + C_4\right) = 0$$

$$\Rightarrow \frac{t^3}{12} + \frac{t^2}{4} - \frac{t^3}{4} - t^2 + C_4 t + \frac{1}{12} - \frac{1}{4} + \frac{t}{4} - t + C_4$$

$$+ \frac{1}{12} - \frac{1}{4} - \frac{t}{4} + C_4 - \frac{t^3}{12} + \frac{t^2}{4} + \frac{t^3}{4} - C_4 = 0$$

$$\Rightarrow -\frac{t^2}{2} + \frac{2}{12} - \frac{1}{2} + 2C_4 - t = 0$$

$$\Rightarrow 2C_4 = \frac{1}{3} + \frac{t^2}{2} + t$$

$$\Rightarrow 2C_4 = \frac{2 + 3t^2 + 6t}{6}$$

$$\Rightarrow C_4 = \frac{3t^2 + 6t + 2}{12}$$

Thus (5) $\Rightarrow G(x,t) = \begin{cases} \frac{x^2}{4} + \frac{1}{2}(1-t)x - t + \frac{3t^2+6t+2}{12}, & -1 \le x < t \\ \frac{x^2}{4} - \frac{1}{2}(1+t)x + \frac{3t^2+6t+2}{12}, & t < x \le 1. \end{cases}$

$$\Rightarrow G(x,t) = \begin{cases} \frac{x^2}{4} + \frac{x}{2} - \frac{xt}{2} - t + \frac{t^2}{4} + \frac{t}{2} + \frac{1}{6}, & -1 \le x \le t. \\ \frac{x^2}{4} - \frac{x}{2} - \frac{tx}{2} + \frac{t^2}{4} + \frac{t}{2} + \frac{1}{6}, & t \le x \le 1. \end{cases}$$

which is our required modified Green's functions.

Thank you 01/01/19.

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Theorem.

Let $G(x, x')$ be the Green's function associated with the problem defined by the equations

$$L(y) = f(x) \quad \text{——— (1)}$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \quad \text{——— (2)}$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0 \quad \text{——— (3)}$$

where L is the operator

$$L \equiv \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)$$

and will also be briefly written as $(py')' + q$.

Then the solution of the problem is

$$y(x) = \int_a^b G(x, x') f(x')\, dx' \quad \text{——— (4)}$$

Proof:

We start with $\int_a^b G(x, x') f(x')\, dx'$ and split the integral w.r.t. x' over the interval $[a, b]$ into two integrals over $[a, x]$ and $(x, b]$.

Hence, we have

$$\int_a^b G(x,x') f(x')\, dx' = \int_a^x G(x,x') f(x')\, dx' + \int_x^b G(x,x') f(x')\, dx'$$

$$= \int_a^{x-0} G(x,x') f(x')\, dx' + \int_{x+0}^b G(x,x') f(x')\, dx'.$$

$$\Rightarrow \int_a^b G(x,x') f(x')\, dx' = I_1 + I_2 \quad \text{——— (5)}$$

Consider

$$I_1 = \int_a^{x-0} G(x,x') f(x')\, dx'$$

From (1)

$$f(x') = (py')' + qy$$

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$$\Rightarrow I_1 = \int_a^{x-0} G(x,x')\{(py')' + qy\}dx'.$$

$$= \int_a^{x-0} G(x,x')(py')'dx' + \int_a^{x-0} G(x,x')qy\,dx'$$

Integrating the 1st integral on R.H.S by parts, we have

$$I_1 = G(x,x')(py')\Big|_{x'=a}^{x'=x-0} - \int_a^{x-0} G'(x,x')(py')dx' + \int_a^{x-0} G(x,x')qy\,dx'$$

$$= G(x,x-0)p(x)y'(x) - G(x,a)p(a)y'(a) - \int_a^{x-0} G'(x,x')(py')dx' + \int_a^{x-0} G(x,x')qy\,dx'.$$

$$= G(x,x-0)p(x)y'(x) - G(x,a)p(a)y'(a) - \left[(pG')y\Big|_a^{x-0} - \int_a^{x-0}(pG')'y\,dx'\right] + \int_a^{x-0} G(x,x')qy\,dx'.$$

$$= G(x,x-0)p(x)y'(x) - G(x,a)p(a)y'(a) - p(x)G'(x,x-0)y(x) + p(a)G'(x,a)y(a) + \int_a^{x-0}\{(pG')' + Gq\}y\,dx'.$$

As $G(x,x')$ satisfies the homogeneous DE (1).

i.e. $L(G) = 0$

$\Rightarrow (pG')' + Gq = 0$ in the interval $[a,x]$,

So

$$I_1 = G(x,x-0)p(x)y'(x) - G(x,a)p(a)y'(a) - p(x)G'(x,x-0)y(x) + p(a)G'(x,a)y(a).$$

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Similarly,

$$I_2 = G(x,b)p(b)y'(b) - G(x,x+0)p(x)y'(x) - G'(x,b)p(b)y(b) + G'(x,x+0)p(x)y(x).$$

Now equation (5) becomes

$$\int_a^b G(x,x')f(x')dx' = G(x,x-0)p(x)y'(x) - G(x,a)p(a)y'(a) - p(x)G'(x,x-0)y(x) + p(a)G'(x,a)y(a) + G(x,b)p(b)y'(b) - G(x,x+0)p(x)y'(x) - G'(x,b)p(b)y(b) + G'(x,x+0)p(x)y(x) \quad \text{---(6)}$$

$G(x,x')$ is continuous on x and x' in the interval $[a,b]$.

$$G(x,x-0) = G(x,x+0).$$

and also, we have

$$G'(x+0,x) - G'(x-0,x) = 1/p(x)$$

$$(6) \Rightarrow \int_a^b G(x,x')f(x')\,dx' = \{G'(x,x+0) - G'(x,x-0)\}p(x)y(x) + \{G(x,x-0) - G(x,x+0)\}p(x)y'(x) + \{G'(x,a)y(a) - G(x,a)y'(a)\}p(a) + \{G(x,b)y'(b) - G'(x,b)y(b)\}p(b)$$

$$\Rightarrow \int_a^b G(x,x')f(x')\,dx' = y(x) + p(a)\{G'(x,a)y(a) - G(x,a)y'(a)\} + p(b)\{G(x,b)y'(b) - G'(x,b)y(b)\}. \quad \text{---(7)}$$

By given boundary conditions given in (2) and (3), we have

$$\alpha_1 G(x,a) + \alpha_2 G'(x,a) = 0$$

$$\beta_1 G(x,b) + \beta_2 G'(x,b) = 0.$$

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$$\Rightarrow G'(x,a) = -\frac{\alpha_1}{\alpha_2} G(x,a)$$

and

$$G'(x,b) = -\frac{\beta_1}{\beta_2} G(x,b).$$

Now equation ⑦ becomes

$$\int_a^b G(x,x')f(x')dx' = y(x) + p(a)\left\{-\frac{\alpha_1}{\alpha_2} G(x,a)\, y(a) - G(x,a)y'(a)\right\} + p(b)\left\{+\frac{\beta_1}{\beta_2} G(x,b)\, y(b) + G(x,b)y'(b)\right\}$$

$$= y(x) + p(a)\left\{-\frac{\alpha_1}{\alpha_2} y(a) - y'(a)\right\} G(x,a) + p(b)\left\{\frac{\beta_1}{\beta_2} y(b) + y'(b)\right\} G(x,b).$$

② and ③ $\Rightarrow$ $y'(a) = -\frac{\alpha_1}{\alpha_2} y(a)$

$$y'(b) = -\frac{\beta_1}{\beta_2} y(b).$$

So $$\int_a^b G(x,x')f(x')dx' = y(x) + p(a)\left\{-\frac{\alpha_1}{\alpha_2} y(a) + \frac{\alpha_1}{\alpha_2} y(a)\right\} G(x,a) + p(b)\left\{\frac{\beta_1}{\beta_2} y(b) - \frac{\beta_1}{\beta_2} y(b)\right\} G(x,b).$$

$$\Rightarrow \int_a^b G(x,x')f(x')dx' = y(x).$$ as required.

Awan Shaul

19-01-19

Chapter end

CHAP 9: Variational Methods

Functional

Let M be the set of functions defined over the interval $[a,b]$. If there is a rule J which assigns each function of M to a real number, then J is called the functional from M to $\mathbb{R}$.

i.e. $J : M \longrightarrow \mathbb{R}$.

Extremal

The curve $y=f(x)$ along which a functional J takes the stationary value is called the extremal.

Stationary Value

The maximum or minimum value of a function or functional is called stationary value.

Euler - Lagrange Equation

Let $I = \int_{x_1}^{x_2} F(x,y,y')\,dx$, where $y=y(x)$ is a continuous function having continuous first and 2nd order derivatives satisfying the following end point conditions.

$$y(x_1) = y_1 \quad , \quad y(x_2) = y_2 .$$

If F is supposed to have continuous first and 2nd order derivatives w.r.t. its arguments then the function $y(x)$ will extremise the given integral if it satisfies the D.E.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 .$$

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Proof.

It is clear that between the fixed points $A(x_1, y_1)$ and $B(x_2, y_2)$ there are infinite number of curves giving infinite number of values to the functional I. We have to find the curve which gives stationary value of the functional.

Let the family of curves passing through fixed points A and B be defined as

$$y(x, \alpha) = y(x, 0) + \alpha \eta(x) \quad \text{——— (1)}$$

where $\eta(x)$ denotes the deviation from the curve $y = y(x) = y(x, 0)$ and α is a labelling parameter for different paths and it is independent of x.

We suppose that the extremal curve corresponds to the value $\alpha = 0$. Since all curves pass through the end points (x_1, y_1) and (x_2, y_2) it follows that $\eta(x_1) = 0 = \eta(x_2)$.

Now $$\frac{\partial I}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_{x_1}^{x_2} F(x, y, y')\, dx.$$

$$= \int_{x_1}^{x_2} \frac{\partial F(x, y, y')}{\partial \alpha}\, dx.$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'} \frac{\partial y'}{\partial \alpha} \right) dx$$

From (1) $\frac{\partial y}{\partial \alpha} = \eta(x)$

Note: y depends on α so y' also depend on α but x is independent of α

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$$y'(x,\alpha) = y'(x,0) + \alpha\eta'(x)$$

$$\frac{\partial y'}{\partial \alpha} = \eta'(x).$$

So $$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x)\right] dx \quad \text{——— (2)}$$

Consider $$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\eta'(x)\,dx = \frac{\partial F}{\partial y'}\eta(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x)dx$$

$$= \frac{\partial F}{\partial y'}\eta(x_2) - \frac{\partial F}{\partial y'}\eta(x_1) - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x)dx$$

$$\Rightarrow \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\eta'(x)\,dx = -\int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x)\,dx$$

Thus equation (2) becomes

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\eta(x) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x)\right] dx.$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)\,dx \quad \text{——— (3)}$$

For the extreme value $\frac{\partial I}{\partial \alpha} = 0$.

$$(3) \Rightarrow \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)\,dx = 0 \quad \text{——— (4)}$$

As $\eta(x)$ is an arbitrary function of x which vanishes at the end points of the interval $[x_1, x_2]$ and also the expression in the square bracket in (4) is a continuous function of x. So by fundamental theorem of calculus of variation, we have

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$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad \text{(5)}$$

which is required Eulel's L. equation.

Special Cases:

① If F is independent of y.

i.e. $\frac{\partial F}{\partial y} = 0$.

so (5) $\Rightarrow$ $0 - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$\Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$\Rightarrow \frac{\partial F}{\partial y'} = \text{constant}.$

② If F is not explicitly dependent on x.

i.e. $\frac{\partial F}{\partial x} = 0$.

As $F = F(x, y, y')$

so $dF = \frac{\partial F}{\partial x}^{0} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial y'} dy'$

$\Rightarrow dF = \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial y'} dy' \qquad \text{(6)}$

From (5) $\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$

$= \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx}$ by chain rule

$\Rightarrow \frac{\partial F}{\partial y} = y' \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)$

$\Rightarrow \frac{\partial F}{\partial y} dy = y' d\left(\frac{\partial F}{\partial y'}\right)$

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⑥ $\Rightarrow dF = y' d\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y'} dy'$

$= d\left(y' \frac{\partial F}{\partial y'}\right)$.

$\Rightarrow dF - d\left(y' \frac{\partial F}{\partial y'}\right) = 0$.

$\Rightarrow d\left(F - y' \frac{\partial F}{\partial y'}\right) = 0$.

$\Rightarrow F - y' \frac{\partial F}{\partial y'} = \text{constant}$.

Special (state and prove) **Fundamental Theorem of the calculus of variations.**

2010, 2015, 2016, 2017

Statement.

If $f(x)$ is continuous in the interval (x_1, x_2) and the integral $\int_{x_1}^{x_2} f(x)\, g(x)\, dx$ is identically zero, where $g(x)$ satisfies the conditions

(i) it is an arbitrary function with continuous derivatives in the interval (x_1, x_2).

(ii) $g(x_1) = 0 = g(x_2)$

then $f(x) = 0 \quad \forall\ x \in [x_1, x_2]$.

Proof.

On contrary suppose that $f(x) \neq 0$ in the interval (x_1, x_2). Then there is at least one point x_0 in the interval (x_1, x_2) such that $f(x_0) \neq 0$.

Since $f(x)$ is continuous in the interval (x_1, x_2) so there exists an interval $(x_0 - \delta, x_0 + \delta)$, $\delta > 0$ surrounding x_0 such that $f(x) > 0 \ \forall\ x \in [x_0 - \delta, x_0 + \delta]$

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Since $g(x)$ is arbitrary so it can be taken as

$$g(x) = \begin{cases} (x-x_0+\delta)^2(x-x_0-\delta)^2 & ; x_0-\delta \le x \le x_0+\delta \\ 0 & ; \text{otherwise} \end{cases}$$

It is clear that $g(x)$ vanishes at the end points of the interval $(x_0-\delta, x_0+\delta)$ and has continuous derivative inside the interval.

Then integral $\int_{x_1}^{x_2} f(x)g(x)dx$ becomes

$$\int_{x_0-\delta}^{x_0+\delta} f(x)(x-x_0+\delta)^2(x-x_0-\delta)^2 dx > 0.$$

which contradicts the assumption that

$$\int_{x_1}^{x_2} f(x)g(x)dx = 0.$$

Hence $f(x) = 0 \quad \forall \; x \in [x_1, x_2]$.

Special

Formulate and Solve the geodesic problem.

To find the curve of minimum length joining two points on a given surface.

Formulation.

Let A and B be any two points on a curve C, lying on a surface S. The distance between the points A and B on any curve $y = y(x)$ is given by

$$S = \int_A^B ds$$

$$= \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

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$$= \int_A^B \sqrt{1 + (dy/dx)^2 + (dz/dx)^2}\, dx.$$

$$= \int_A^B \sqrt{1 + y'^2 + z'^2}\, dx$$

Here ds is the element of arc along the curve. When the points A and B lie in the XY-plane, then the expression for S becomes

$$S = \int_A^B \sqrt{1 + y'^2}\, dx$$

A/2018. use calculus of variation to prove that a straight line is the shortest distance b/w two pts in plane

Solution.

Here $F = \sqrt{1+y'^2}$. So $I = \int_a^b \sqrt{1+y'^2}\, dx$, Subject to the end point conditions $y(a) = y_0$ and $y(b) = y_1$, will be minimum if $y = y(x)$ satisfies the Eulel-Lagrange equation

$$\frac{\partial F}{\partial y'} = c \quad \text{(any constant)}$$

$$\Rightarrow \frac{\partial}{\partial y'} \sqrt{1+y'^2} = c$$

$$\Rightarrow \frac{1}{2}(1+y'^2)^{-1/2} 2y' = c$$

$$\Rightarrow \frac{y'}{\sqrt{1+y'^2}} = c$$

Squaring on both sides, we have

$$\frac{y'^2}{1+y'^2} = c^2$$

$$\Rightarrow y'^2 = c^2(1+y'^2)$$

$$\Rightarrow y'^2 - c^2 y'^2 = c^2$$

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$$\Rightarrow y'^2(1-c^2) = c^2$$

$$\Rightarrow y'^2 = \frac{c^2}{1-c^2} = a^2 \text{ (say)}$$

$$\Rightarrow y' = \pm a$$

$$\Rightarrow \frac{dy}{dx} = \pm a$$

$$\Rightarrow dy = \pm a\,dx \qquad \Rightarrow y = \pm ax + b$$

which is the equation of straight line.

Example.

Give geometrical interpretation of the variational problem

$$\int \sqrt{1+y'^2}\,dx \qquad \text{——— (A)}$$

with the boundary conditions $y(0)=0$ and $y(1)=1$.

Solve the problem for extremal. Find the stationary value of the integral and compare it with the values of the integral which are obtained for curves that join the same end points but are different from extremal.

Solution.

Geometrical interpretation.

The problem is to find the curve of shortest length joining the points $(0, 0)$ and $(1, 1)$.

Sol.

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Here $F = \sqrt{1+y'^2}$ ——— (1)

For finding the curve of shortest length, F must has to satisfies Euler-Lagrange equation

i.e. $\frac{\partial F}{\partial y'} = c$.

(1) $\Rightarrow \frac{\partial \sqrt{1+y'^2}}{\partial y'} = c \Rightarrow \frac{1}{2}(1+y'^2)^{-1/2}(2y') = c$.

$\Rightarrow \frac{y'}{\sqrt{1+y'^2}} = c$

Squaring on both sides, we get

$\frac{y'^2}{1+y'^2} = c^2 \Rightarrow y'^2 = c^2(1+y'^2)$

$\Rightarrow y'^2 - c^2y'^2 = c^2 \Rightarrow y'^2(1-c^2) = c^2$

$\Rightarrow y'^2 = \frac{c^2}{1-c^2} = a^2$

$\Rightarrow y' = \pm a \Rightarrow \frac{dy}{dx} = \pm a$.

$\Rightarrow dy = \pm a\,dx$.

$\Rightarrow y = \pm ax + b$. ——— (2)

Applying boundary conditions, we have

$y(0) = 0 \Rightarrow b = 0$

$y(1) = 1 \Rightarrow \pm a = 1. \Rightarrow a = \pm 1$.

(2) $\Rightarrow y = x$ is the required curve.

For stationary value we have to put $y = x$ in equation (A) to get

Stationary value $= \int_0^1 \sqrt{1+y'^2}\,dx$.

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As $y = x$ so $y' = 1$.

Stationary value $= \int_0^1 \sqrt{1 + (1)^2}\, dx$

$= \sqrt{2} \int_0^1 dx$.

$= \sqrt{2}\, x \Big|_0^1 = \sqrt{2}$.

Now we have to find the value of integral in (A), for the path different from $y = x$ which passes through $(0,0)$ and $(1,1)$.

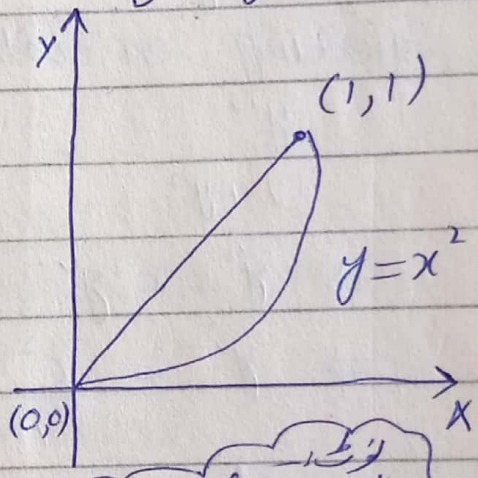
For this, we use the path along $y = x^2$.

$$I = \int_0^1 \sqrt{1 + y'^2}\, dx$$

$$= \int_0^1 \sqrt{1 + (2x)^2}\, dx.$$

$$= \int_0^1 \sqrt{1 + 4x^2}\, dx$$

$$= \int_0^1 \sqrt{4\left(\frac{1}{4} + x^2\right)}\, dx.$$



$$\Rightarrow I = 2\int_0^1 \sqrt{x^2 + \left(\frac{1}{2}\right)^2}\, dx \quad \text{---} (3)$$

As we know that

$$\int \sqrt{x^2 + a^2}\, dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2} \ln\left(x + \sqrt{x^2+a^2}\right)$$

So (3) $\Rightarrow I = 2\left[\frac{x\sqrt{x^2 + \frac{1}{4}}}{2} + \frac{1}{2}\cdot\frac{1}{4}\ln\left(x + \sqrt{x^2 + \frac{1}{4}}\right)\right]_0^1$

$$= \left[x\sqrt{x^2 + \frac{1}{4}} + \frac{1}{4}\ln\left(x + \sqrt{x^2 + \frac{1}{4}}\right)\right]_0^1$$

$$= \sqrt{1 + \frac{1}{4}} + \frac{1}{4}\ln\left(1 + \sqrt{1 + \frac{1}{4}}\right) - 0 - \frac{1}{4}\ln\left(\sqrt{1/4}\right)$$

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$$= \frac{\sqrt{5}}{2} + \frac{1}{4}\ln\left(1+\frac{\sqrt{5}}{2}\right) - \frac{1}{4}\ln\left(\frac{1}{2}\right).$$

$$= \frac{\sqrt{5}}{2} + \frac{1}{4}\ln\left(1+\frac{\sqrt{5}}{2}\right)/\frac{1}{2}$$

$$= \frac{\sqrt{5}}{2} + \frac{1}{4}\ln(2+\sqrt{5}).$$

This value is always greater than $\sqrt{2}$ which shows that $\sqrt{2}$ is the stationary value

Special

Formulate and Solve the Brachistochrone Problem

2010
2013 To find the path of quickest descent, joining two points in space, for a particle moving under gravity.

Formulation.

We know that the time dt and displacement ds are related to the instantaneous velocity $\vec{v}$ by $v = ds/dt$.

Therefore, $\bar{t} = \int_A^B dt$

where $\bar{t}$ denotes the total time taken.

$$\Rightarrow \bar{t} = \int_A^B \frac{ds}{v} \qquad \text{---------- (1)}$$

For the falling particle

$$K.E. = P.E.$$

$$\Rightarrow \frac{1}{2}mv^2 = mgy$$

Vertically $h = y$

$$\Rightarrow v^2 = 2gy$$

$$\Rightarrow v = \sqrt{2gy}$$

Now equation (1) becomes

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$$I = \int_A^B \frac{ds}{\sqrt{2gy}} = \int_A^B \frac{1}{\sqrt{2gy}}\, ds$$

$$= \int_A^B \frac{1}{\sqrt{2gy}} \sqrt{1+y'^2}\, dx.$$

in xy plane $ds = \sqrt{1+y'^2}\, dx$

$$\Rightarrow I = \frac{1}{\sqrt{2g}} \int_A^B \sqrt{\frac{1+y'^2}{y}}\, dx \quad \text{——— (2)}$$

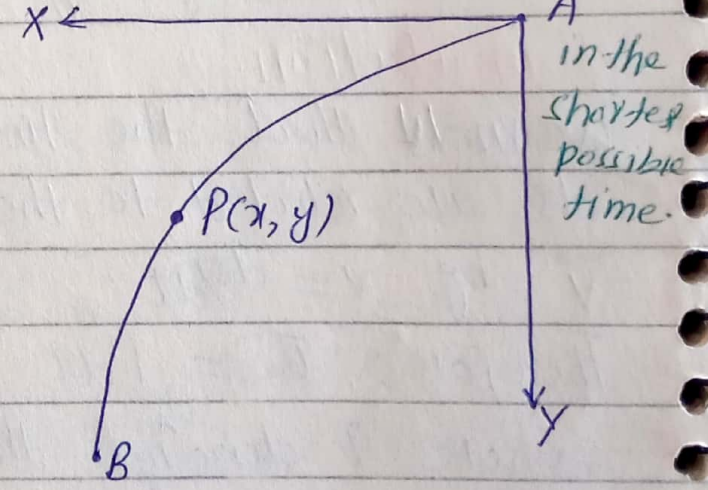
which is the form of a functional.

By minimizing this functional we can find the required time A/2018 Find the equation of a path in space down which a particle will fall from one pt to another in the shortest possible time.

Solution.

Let a particle fall from a point A to another point B.

There are infinite number of paths between A and B, but we consides the only path along which the time is taken minimum.



Let P(x, y) be the position of the particle at time t as shown in Fig.

Let us denote the arc element of the curve $y = y(x)$ by ds, then the total time taken by the particle falling from A to B is given by

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$$I = \int_A^B dt = \frac{1}{\sqrt{2g}} \int_A^B \sqrt{\frac{1+y'^2}{y}}\, dx \quad \because \text{using } (2)$$

we have to find the extremal curve through A and B. Therefore, it must satisfy the Euler-Lagrange equation

i.e. $F - y' \frac{\partial F}{\partial y'} = C$ ——— (3)

F does not depend explicitly on x

Here $F = \sqrt{\frac{1+y'^2}{y}}$.

Therefore (3) $\Rightarrow \sqrt{\frac{1+y'^2}{y}} - y' \cdot \frac{\partial}{\partial y'} \sqrt{\frac{1+y'^2}{y}} = C.$

$$\Rightarrow \sqrt{\frac{1+y'^2}{y}} - y' \frac{1}{\sqrt{y}} \frac{\partial}{\partial y'} \sqrt{1+y'^2} = C.$$

$$\Rightarrow \sqrt{\frac{1+y'^2}{y}} - \frac{y'}{\sqrt{y}} \frac{y'}{\sqrt{1+y'^2}} = C.$$

$$\Rightarrow \frac{1+y'^2 - y'^2}{\sqrt{y}\sqrt{1+y'^2}} = C$$

$$\Rightarrow \frac{1}{\sqrt{y(1+y'^2)}} = C$$

Taking square on both sides, we have

$$\frac{1}{y(1+y'^2)} = C^2$$

$$\Rightarrow 1 = y(1+y'^2)C^2.$$

$$\Rightarrow \frac{1}{C^2 y} = 1 + y'^2$$

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$$\Rightarrow y'^2 = \frac{1}{c^2y} - 1 = \frac{1-c^2y}{c^2y}$$

$$\Rightarrow y' = \sqrt{\frac{1-c^2y}{c^2y}} \Rightarrow \frac{dy}{dx} = \sqrt{\frac{1-c^2y}{c^2y}}$$

$$\Rightarrow dy = \sqrt{\frac{1-c^2y}{c^2y}}\, dx.$$

$$\text{or } dx = \sqrt{\frac{c^2y}{1-c^2y}}\, dy \quad \text{---------- (4)}$$

Let us substitute $c^2y = \sin^2\theta/2$

$$\Rightarrow y = \frac{1}{c^2}\sin^2\theta/2.$$

$$\Rightarrow dy = \frac{1}{c^2} 2\sin\theta/2 \cos\theta/2 \cdot \frac{1}{2} d\theta.$$

$$\Rightarrow dy = \frac{1}{2c^2}\sin\theta\, d\theta.$$

Now equation (4) becomes

$$dx = \sqrt{\frac{\sin^2\theta/2}{1-\sin^2\theta/2}} \cdot \frac{1}{2c^2}\sin\theta\, d\theta.$$

$$= \frac{1}{2c^2}\sqrt{\frac{\sin^2\theta/2}{\cos^2\theta/2}}\, 2\sin\theta/2 \cos\theta/2\, d\theta.$$

$$= \frac{1}{2c^2}\frac{\sin\theta/2}{\cos\theta/2} 2\sin\theta/2 \cos\theta/2\, d\theta$$

$$\Rightarrow dx = \frac{1}{c^2}\sin^2\theta/2\, d\theta$$

Integrating, we get

$$x = \frac{1}{c^2}\int \sin^2\theta/2\, d\theta + c_1$$

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$$= \frac{1}{c^2}\int \left(\frac{1-\cos\theta}{2}\right) d\theta + c_1 = \frac{1}{2c^2}(\theta - \sin\theta) + c_1$$

$$\Rightarrow x = \frac{1}{2c^2}(\theta - \sin\theta) + c_1 \quad \text{——— (5)}$$

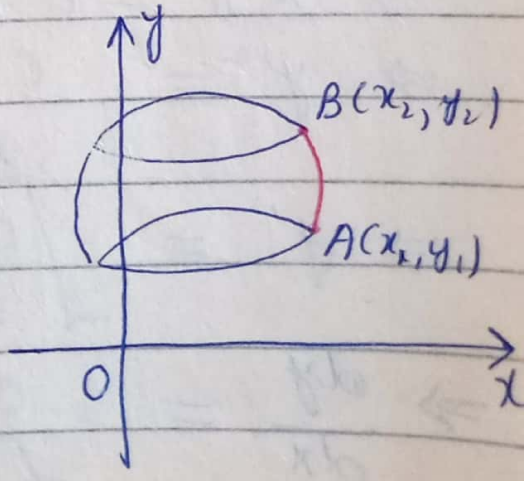
and $y = \frac{1}{c^2}\sin^2\theta/2 = \frac{1}{c^2}\left(\frac{1-\cos\theta}{2}\right)$

$$\Rightarrow y = \frac{1}{2c^2}(1-\cos\theta) \quad \text{——— (6)}$$

Equation (5) and (6) are the parametric equations of the cycloid.
The constants c and c_1 can be determined from the end point conditions.
Therefore, the curve down which the particle takes the minimum time is cycloid.

Example 1. V.V. Imp.
Find the curve joining the points (x_1, y_1) and (x_2, y_2) which gives minimum area of surface of revolution generated around (i) y-axis (ii) x-axis.

~~Example 1~~ Solution.
Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be points in xy-plane. We have to find a curve which gives the minimum area of the surface of revolution.



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(i)

Let us assume that the curve revolves about the y-axis. In this case, area of the surface of revolution will be given by.

$$\text{Area} = \int_A^B 2\pi x \, ds$$

$$= 2\pi \int_A^B x \sqrt{1+y'^2} \, dx.$$

For the minimum value it must satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial y'} = C$$

$$\Rightarrow \frac{\partial}{\partial y'} x\sqrt{1+y'^2} = C$$

$$\Rightarrow x \frac{y'}{\sqrt{1+y'^2}} = C \Rightarrow \frac{xy'}{\sqrt{1+y'^2}} = C$$

Squaring on both sides, we have

$$\frac{x^2 y'^2}{1+y'^2} = C^2$$

$$\Rightarrow x^2 y'^2 = C^2(1+y'^2)$$

$$\Rightarrow x^2 y'^2 - C^2 y'^2 = C^2$$

$$\Rightarrow y'^2 = \frac{C^2}{x^2 - C^2}$$

$$\Rightarrow y' = \sqrt{\frac{C^2}{x^2 - C^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{C}{\sqrt{x^2 - C^2}}$$

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$$\Rightarrow dy = \frac{c}{\sqrt{x^2-c^2}}\,dx \Rightarrow y = c\int \frac{dx}{\sqrt{x^2-c^2}}.$$

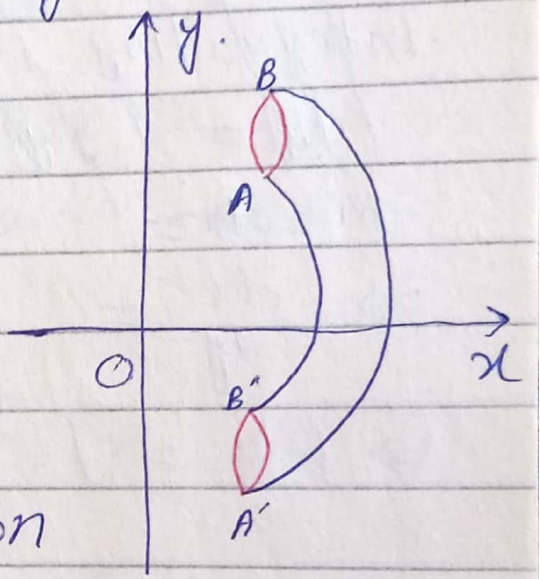
$$\Rightarrow y = c\cosh^{-1}\left(\frac{x}{c}\right) + c_1$$

(ii)

Let us assume that the curve revolves about the x-axis. In this case, area of the surface of revolution will be given by

$$\text{Area} = \int_A^B 2\pi y\,ds$$

$$= 2\pi\int_A^B y\sqrt{1+y'^2}\,dx$$



For the minimum value it must satisfy the Euler-Lagrange equation

$$F - y'\frac{\partial F}{\partial y'} = c.$$

$$\Rightarrow y\sqrt{1+y'^2} - y'\cdot\frac{\partial}{\partial y'}\, y\sqrt{1+y'^2} = c.$$

$$\Rightarrow y\sqrt{1+y'^2} - y'y\cdot\frac{y'}{\sqrt{1+y'^2}} = c.$$

$$\Rightarrow \frac{y(1+y'^2) - yy'^2}{\sqrt{1+y'^2}} = c$$

$$\Rightarrow \frac{y + yy'^2 - yy'^2}{\sqrt{1+y'^2}} = c$$

$$\Rightarrow \frac{y}{\sqrt{1+y'^2}} = c$$

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Taking square on both sides, we have

$$\frac{y^2}{1+y'^2} = c^2 \Rightarrow y^2 = c^2(1+y'^2)$$

$$\Rightarrow y^2 - c^2 = c^2 y'^2 \Rightarrow y' = \frac{\sqrt{y^2-c^2}}{c}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c}$$

$$\Rightarrow dy = \sqrt{y^2-c^2}\, dx.$$

Integrating, we get

$$\int dy = \int \sqrt{y^2-c^2}\, dx$$

or ~~dx~~

$$\Rightarrow \frac{dx}{dy} = \frac{c}{\sqrt{y^2-c^2}}$$

$$\Rightarrow \int dx = c \int \frac{dy}{\sqrt{y^2-c^2}}$$

$$\Rightarrow x = c \cosh^{-1}\left(\frac{y}{c}\right) + c_2.$$

Example 2.

On what curves can the functional $I = \int_0^{\pi/2} (y'^2 - y^2)\, dx$ with the end point conditions $y(0) = 0$, $y(\pi/2) = 1$ be extremized.

Solution.

Here $F = y'^2 - y^2$.

The Euler-Lagrange equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$-2y - \frac{d}{dx}(2y') = 0.$$

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$\Rightarrow 2y + 2\frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$

$\Rightarrow y + \frac{d^2y}{dx^2} = 0 \quad \Rightarrow y + y'' = 0$

$\Rightarrow D^2y + y = 0$

The characteristic equation is

$D^2 + 1 = 0$

$D^2 = -1 \Rightarrow D = \pm i$ (complex)

The general solution is

$y(x) = C_1 \cos x + C_2 \sin x$ ——— (1)

Applying boundary condition, we have

$y(0) = 0 \Rightarrow \boxed{C_1 = 0}$

(1) $\Rightarrow y(x) = C_2 \sin x.$

$y\left(\frac{\pi}{2}\right) = 1 \Rightarrow C_2 \sin\left(\frac{\pi}{2}\right) = 1 \Rightarrow \boxed{C_2 = 1}$

Hence the required extremal is

$y = \sin x.$

Exercise

Question 1.

From among the curves connecting the points A(1, 3) and B(2, 5) find the extremal curve of the functional.

$$I[y] = \int_1^2 y'(1 + x^2y')\,dx.$$

Solution.

For the extremal F must satisfy the Euler-Lagrange equation

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$$\frac{\partial F}{\partial y'} = C$$

$$\Rightarrow \frac{\partial}{\partial y'} y'(1+x^2y') = C$$

$$\Rightarrow y'(x^2) + 1 + x^2y' = C$$

$$\Rightarrow 2x^2y' + 1 = C$$

$$\Rightarrow 2x^2y' = C - 1$$

$$\Rightarrow y' = \frac{C-1}{2x^2}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{C-1}{2}\left(\frac{1}{x^2}\right)$$

$$\Rightarrow y = \frac{C-1}{2}\int \frac{1}{x^2}\,dx + C_1$$

$$\Rightarrow y = -\frac{(C-1)}{2}\cdot\frac{1}{x} + C_1 \quad \text{——— (1)}$$

$y(1)=3 \Rightarrow -\frac{(C-1)}{2} + C_1 = 3$

$$\Rightarrow \frac{1-C}{2} = 3 - C_1$$

$$\Rightarrow 1 - C = 6 - 2C_1$$

$$\Rightarrow 2C_1 - C = 5 \quad \text{——— (2)}$$

$y(2)=5 \Rightarrow \frac{1-C}{2}\cdot\frac{1}{2} + C_1 = 5$

$$\frac{1-C}{4} = 5 - C_1$$

$$1 - C = 20 - 4C_1$$

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$\Rightarrow 4C_1 - C = 19$ ——— (3)

(2) − (1) $\Rightarrow 4C_1 - 2C_1 = 19 - 5$

$\Rightarrow 2C_1 = 14$

$\Rightarrow \boxed{C_1 = 7}$

(2) $\Rightarrow 14 - C = 5 \quad \Rightarrow \boxed{C = 9}$

Thus,

(1) $\Rightarrow y = -\frac{(9-1)}{2} \cdot \frac{1}{x} + 7$

$\Rightarrow y = 7 - 4/x$.

Question 2.

Find the extremal of the problem

$I[y] = \int_0^1 (y'^2 + 3y + 2x)\, dx$.

$y(0) = 0, \quad y(1) = 1$.

Solution.

Here $F = y'^2 + 3y + 2x$. For the extremal F must satisfy the E-Lagrange equation.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$\Rightarrow 3 - \frac{d}{dx}(2y') = 0$

$\Rightarrow 3 - 2y'' = 0 \quad \Rightarrow y'' = \frac{3}{2}$

$\Rightarrow y' = \frac{3}{2}x + C_1$

$\Rightarrow y = \frac{3}{4}x^2 + C_1 x + C_2$ ——— (1)

$y(0) = 0 \Rightarrow \boxed{0 = C_2}$

$y(1) = 1 \Rightarrow 1 = \frac{3}{4} + C_1 \quad \Rightarrow C_1 = 1 - \frac{3}{4} = 1/4$

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① $\Rightarrow y = \frac{3}{4}x^2 + \frac{1}{4}x$.

Question 3: 2015,

Find extremals of the following functional subject to the given conditions

$$I[y] = \int_0^1 (y^2 + y'^2)\,dx \;, \; y(0)=0, \; y(1)=1$$

Solution.

Here $F = y^2 + y'^2$. For the extremal F must satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$2y - \frac{d}{dx}(2y') = 0 \quad \Rightarrow y - y'' = 0$$

$$\Rightarrow y'' - y = 0$$

The char. equation is

$$D^2 - 1 = 0 \quad \Rightarrow D^2 = 1 \quad \Rightarrow D = \pm 1$$

$$\Rightarrow y(x) = C_1 e^x + C_2 e^{-x} \qquad \text{——— ①}$$

$$y(0) = 0 \Rightarrow 0 = C_1 + C_2$$

$$\Rightarrow \boxed{C_1 = -C_2}$$

$$y(1) = 1 \Rightarrow 1 = C_1 e^1 + C_2 e^{-1}$$

$$\Rightarrow 1 = -C_2 e + C_2 e^{-1}$$

$$\Rightarrow 1 = C_2(e^{-1} - e^1)$$

$$\Rightarrow C_2 = \frac{1}{e^{-1} - e^1}$$

$$\Rightarrow C_1 = \frac{-1}{e^{-1} - e^1}$$

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$$① \Rightarrow y(x) = \frac{1}{e^1 - e^{-1}} e^x - \frac{1}{e^1 - e^{-1}} e^{-x}$$

$$\Rightarrow y(x) = \frac{e^x - e^{-x}}{e^1 - e^{-1}}.$$

$$\Rightarrow y(x) = \frac{(e^x - e^{-x})/2}{(e^1 - e^{-1})/2} = \frac{\sinh x}{\sinh(1)}$$

Question 4:

Find the extremals of the following functional subject to the given conditions.

$$I[y] = \int_0^1 (y'^2 + 4y^2)\, dx.$$

$$y(0) = e^2 \; ; \quad y(1) = 1.$$

Solution.

Here $F = y'^2 + 4y^2$. For the extremal F must satisfy the Euler-Lagrange equation.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0.$$

$$8y - \frac{d}{dx}(2y') = 0$$

$$\Rightarrow 8y - 2y'' = 0$$

$$\Rightarrow 4y - y'' = 0$$

$$\Rightarrow y'' - 4y = 0$$

The aux. equation is

$$D^2 - 4 = 0 \Rightarrow D^2 = 4 \Rightarrow D = \pm 2$$

$$\Rightarrow y(x) = C_1 e^{2x} + C_2 e^{-2x} \quad \text{——— (1)}$$

$$y(0) = e^2 \Rightarrow e^2 = C_1 + C_2 \quad \text{——— (2)}$$

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$y(1)=1 \Rightarrow 1 = C_1 e^2 + C_2 e^{-2}$ ——— ③

② $\Rightarrow C_1 = e^2 - C_2$

③ $\Rightarrow 1 = (e^2 - C_2)e^2 + C_2 e^{-2}$

$= e^4 - C_2 e^2 + C_2 e^{-2}$

$= e^4 + C_2(e^{-2} - e^2)$

$\Rightarrow 1 - e^4 = C_2(e^{-2} - e^2)$

$\Rightarrow C_2 = \frac{1-e^4}{e^{-2}-e^2} = \frac{e^2(e^{-2}-e^2)}{e^{-2}-e^2}$

$\Rightarrow \boxed{C_2 = e^2}$

② $\Rightarrow \boxed{C_1 = 0}$

① $\Rightarrow y(x) = e^2 e^{-2x}$.

Question 5.

Find the extremal of the following functional subject to the given conditions. $\pi/2$

$I[y] = \int_0 (y^2 - y'^2 - 8y\cosh x)\,dx$.

$y(0) = 2,\ y(\frac{\pi}{2}) = 2\cosh(\frac{\pi}{2})$.

Solution.

Here $F = y^2 - y'^2 - 8y\cosh x$. For the extremal F must satisfy the Euler-Lagrange equation.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

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$$2y - 8\cosh x - \frac{d}{dx}(-2y') = 0$$
$$\Rightarrow 2y - 8\cosh x + 2y'' = 0$$
$$\Rightarrow y'' + y - 4\cosh x = 0$$
$$\Rightarrow y'' + y = 4\cosh x.$$

For homogeneous part

$$y'' + y = 0$$

The auxi equation is

$$D^2 + 1 = 0 \Rightarrow D = \pm i$$

$$y_c(x) = C_1 \cos x + C_2 \sin x \quad \text{——— (1)}$$

For particulas solution

$$y_p = \frac{1}{D^2+1} 4\cosh x = \frac{1}{D^2+1} \cdot 2(e^x + e^{-x})$$

$$= \frac{2e^x}{D^2+1} + \frac{2e^{-x}}{D^2+1}$$

$$= \frac{xe^x}{2} + \frac{xe^{-x}}{2} = e^x + e^{-x}$$

The general solution is

$$y(x) = y_c(x) + y_p(x)$$
$$y(x) = C_1 \cos x + C_2 \sin x + e^x + e^{-x}$$
$$y(0) = 2 \Rightarrow 2 = C_1 + 1 + 1$$
$$\Rightarrow \boxed{C_1 = 0}$$
$$y\left(\frac{\pi}{2}\right) = 2\cosh\left(\frac{\pi}{2}\right)$$
$$\Rightarrow 2\cosh\left(\frac{\pi}{2}\right) = C_2 \sin(\pi/2) + e^{\pi/2} + e^{-\pi/2}$$
$$\Rightarrow 2\cosh(\pi/2) = C_2 + e^{\pi/2} + e^{-\pi/2}$$

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$\Rightarrow 2\cosh(\pi/2) = C_2 + 2\cosh(\pi/2)$.

$\Rightarrow \boxed{C_2 = 0}$

Hence $y(x) = e^x + e^{-x} = 2\cosh x$.

Question 6. 2009, 2010, 2011.

Find the stationary values for the integral $\int_{A(0,\pi)}^{B(\pi,0)} (2y\sin x + y'^2)\,dx$.

Solution.

Here $F = 2y\sin x + y'^2$. For the extremal the F must satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0.$$

$$2\sin x - \frac{d}{dx}(2y') = 0$$

$\Rightarrow \sin x - y'' = 0$

$\Rightarrow y'' = \sin x$

$\Rightarrow y' = -\cos x + C_1$

$\Rightarrow y(x) = -\sin x + C_1 x + C_2$ ——— (1)

$y(0) = \pi \Rightarrow \pi = -0 + C_2$

$\Rightarrow C_2 = \pi + 0$

$\Rightarrow \boxed{C_2 = \pi}$

$y(\pi) = 0 \Rightarrow 0 = C_1\pi + \pi$

$\Rightarrow C_1\pi = -\pi$

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$\Rightarrow \boxed{C_1 = -1}$

Thus ① $\Rightarrow$ $y(x) = -\sin x - x + \pi$

Question 7. 2009, 2010.

Find the extremal for the function

$$I[y] = \int_{A(0,0)}^{B(\frac{\pi}{2}, \frac{\pi}{2})} (y'^2 + 2xy - y^2)\, dx.$$

Solution.

Here $F = y'^2 + 2xy - y^2$. For the extremal F must satisfy the Euler-Lagrange equation.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0.$$

$$2x - 2y - \frac{d}{dx}(2y) = 0$$

$$\Rightarrow x - y - y'' = 0$$

$$\Rightarrow y'' + y = x$$

For homogeneous part

$$y'' + y = 0$$

The aux. equation is

$$D^2 + 1 = 0 \quad \Rightarrow D^2 = -1 \Rightarrow D = \pm i$$

$$\Rightarrow y_c(x) = C_1 \cos x + C_2 \sin x.$$

For particular integral

$$y_p = \frac{1}{D^2+1} x.$$

$$= (1+D^2)^{-1} x = (1 + D^2 + \cdots) x$$

$$\Rightarrow y_p = x$$

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The general solution is

$y(x) = y_c(x) + y_p(x)$

$= C_1 \cos x + C_2 \sin x + x$ ——— (1)

$y(0) = 0 \Rightarrow \boxed{0 = C_1}$

$y(\pi/2) = \pi/2$

$\Rightarrow \frac{\pi}{2} = C_2 + \frac{\pi}{2} \Rightarrow \boxed{C_2 = 0}$

Thus (1) $\Rightarrow y(x) = x$.

Question 8: 2014,

Minimize the integral $I[y] = \int (y'^2 + x^2) dx$ subject to the conditions $y(0) = 1$, $y(1) = 0$.

Solution.

Here $F = y'^2 + x^2$. The integral will be minimum if $y = y(x)$ satisfies the Eules-Lagrange equation

$$\frac{\partial F}{\partial y'} = C$$

$\Rightarrow 2y' = C \Rightarrow \frac{dy}{dx} = \frac{C}{2}$

$\Rightarrow y = \frac{C}{2}x + C_1$ ——— (1)

$y(0) = 1 \Rightarrow \boxed{1 = C_1}$

$y(1) = 0 \Rightarrow 0 = \frac{C}{2} + 1$

$\Rightarrow \frac{C}{2} = -1 \Rightarrow \boxed{C = -2}$

(1) $\Rightarrow y = -x + 1$. or $y = 1 - x$.

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Euler-Lagrange equation for n dependent variables. 2011, 2014,

Let $y_1, y_2, y_3, \cdots y_n$ be n dependent functions with $y_1, y_2, y_3, \cdots y_n$ held fixed at x_1, x_2. Then the functions $y_1, y_2, y_3, \cdots, y_n$ may extremize the integral

$$I = \int_{x_1}^{x_2} F(x, y_1, y_2, y_3, \cdots y_n, y_1', y_2', y_3', \cdots, y_n')\, dx$$

with boundary conditions

$$y_k(x_1) = \text{constant}, \quad y_k(x_2) = \text{constant}$$

where $k = 1, 2, 3, \cdots n$.

It is necessary that functions $y_1, y_2, y_3, \cdots y_n$ satisfy the following,

$$\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right) = 0; \quad k = 1, 2, 3, \cdots n.$$

Proof.

The variation of I as δI when y_k varies by δy_k, $k = 1, 2, 3 \cdots n$ is given by

$$\delta I = \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y_1}\delta y_1 + \frac{\partial F}{\partial y_2}\delta y_2 + \cdots + \frac{\partial F}{\partial y_n}\delta y_n\right) + \left(\frac{\partial F}{\partial y_1'}\delta y_1' + \frac{\partial F}{\partial y_2'}\delta y_2' + \cdots + \frac{\partial F}{\partial y_n'}\delta y_n'\right)\right] dx$$

$$= \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y_1}\delta y_1 + \frac{\partial F}{\partial y_1'}\delta y_1'\right) + \left(\frac{\partial F}{\partial y_2}\delta y_2 + \frac{\partial F}{\partial y_2'}\delta y_2'\right) + \cdots + \left(\frac{\partial F}{\partial y_n}\delta y_n + \frac{\partial F}{\partial y_n'}\delta y_n'\right)\right] dx. \quad \text{---} (1)$$

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Consider

$$\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y_k'}\delta y_k'\right)dx = \frac{\partial F}{\partial y_k'}\delta y_k\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2}\delta y_k \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right)dx.$$

Since y_k = Constant at x_1, x_2.

Therefore, $\delta y_k = 0$ at x_1, x_2

$$\Rightarrow \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y_k'}\delta y_k'\right)dx = \frac{\partial F}{\partial y_k'}\cancel{\delta y_k}^{0}\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2}\delta y_k \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right)dx$$

$$= -\int_{x_1}^{x_2}\delta y_k \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right)dx.$$

Thus

$$\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y_k}\delta y_k + \frac{\partial F}{\partial y_k'}\delta y_k'\right)dx = \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y_k}\delta y_k - \delta y_k \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right)\right]dx$$

$$= \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right)\right)\delta y_k\, dx \qquad \text{——— (2)}$$

where $k = 1, 2, 3, \cdots n$.

Subsituting (2) in (1), we get

$$\delta I = \int_{x_1}^{x_2}\left[\left(\frac{\partial F}{\partial y_1} - \frac{d}{dx}\frac{\partial F}{\partial y_1'}\right)\delta y_1 + \left(\frac{\partial F}{\partial y_2} - \frac{d}{dx}\frac{\partial F}{\partial y_2'}\right)\delta y_2 + \cdots + \left(\frac{\partial F}{\partial y_n} - \frac{d}{dx}\frac{\partial F}{\partial y_n'}\right)\delta y_n\right]dx.$$

For extremal $\delta I = 0$, therefore.

$$\int_{x_1}^{x_2}\sum_{k=1}^{n}\left[\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right)\right]\delta y_k\, dx = 0$$

By the fundamental theorem of Calculus

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of variation, we have

$$\left[\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right)\right]\delta y_k = 0 \quad ; k = 1,2,3,\cdots n$$

Since $\delta y_k \neq 0$, $k = 1,2,3,\cdots n$ so, we have

$$\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right) = 0 \quad ; k = 1,2,3,\cdots, n.$$

which are the required n. Eules-Lagrange equations.

Example 1. special

Find the extremal for $I = \int_0^{\pi/2} (y'^2 + z'^2 + 2yz)\,dx$ where the boundary conditions are

$y(0) = 0$, $y(\pi/2) = 1$; $z(0) = 0$, $z(\pi/2) = -1$.

Solution.

Here $F = y'^2 + z'^2 + 2yz$. For the extremal F must satisfy the Eules-Lagrange equations

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{——— (1)}$$

and $$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0 \quad \text{——— (2)}$$

$$(1) \Rightarrow 2z - \frac{d}{dx}(2y') = 0$$

$$\Rightarrow z - y'' = 0 \quad \text{——— (3)}$$

$$(2) \Rightarrow 2y - \frac{d}{dx}(2z') = 0$$

$$\Rightarrow y - z'' = 0 \quad \text{——— (4)}$$

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③ $\Rightarrow z = y'' \Rightarrow z' = y'''$

$\Rightarrow z'' = y^{(4)}$

④ $\Rightarrow y - y^{(4)} = 0.$

$\Rightarrow \frac{d^4}{dx^4} y - y = 0$

$\Rightarrow D^4 y - y = 0 \quad \because D^4 = d^4/dx^2$

The charac. equ. is

$D^4 - 1 = 0 \Rightarrow (D^2-1)(D^2+1) = 0$

$\Rightarrow (D-1)(D+1)(D^2+1) = 0.$

$\Rightarrow D = -1, 1, \pm i$

The general solution is

$$y = C_1 \cos x + C_2 \sin x + C_3 e^{x} + C_4 e^{-x} \quad \text{——⑤}$$

$$\Rightarrow y' = -C_1 \sin x + C_2 \cos x + C_3 e^{x} - C_4 e^{-x}$$

$$\Rightarrow y'' = -C_1 \cos x - C_2 \sin x + C_3 e^{x} + C_4 e^{-x}$$

③ $\Rightarrow$

$$z = -C_1 \cos x - C_2 \sin x + C_3 e^{x} + C_4 e^{-x} \quad \text{——⑥}$$

Applying given boundary conditions, we get

⑤ $\Rightarrow y(0) = C_1 + C_3 + C_4 = 0$

$\Rightarrow C_1 + C_3 + C_4 = 0 \quad \text{——⑦}$

and $y(\frac{\pi}{2}) = C_2 + C_3 e^{\pi/2} + C_4 e^{-\pi/2} = 1$

$\Rightarrow C_2 + C_3 e^{\pi/2} + C_4 e^{-\pi/2} = 1 \quad \text{——⑧}$

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Now ⑥ $\Rightarrow z(0) = -C_1 + C_3 + C_4 = 0$

$\Rightarrow -C_1 + C_3 + C_4 = 0$ ——— ⑨

and $z(\pi/2) = -C_2 + C_3 e^{\pi/2} + C_4 e^{-\pi/2} = -1$

$\Rightarrow -C_2 + C_3 e^{\pi/2} + C_4 e^{-\pi/2} = -1$ ——— ⑩

Adding ⑦ and ⑨, we get

$2C_3 + 2C_4 = 0 \Rightarrow \boxed{C_3 = -C_4}$

Adding ⑧ and ⑩, we get

$2C_3 e^{\pi/2} + 2C_4 e^{-\pi/2} = 0$

$\Rightarrow C_3 e^{\pi/2} + C_4 e^{-\pi/2} = 0$

$\Rightarrow -C_4 e^{\pi/2} + C_4 e^{-\pi/2} = 0 \quad \because C_3 = -C_4$

$\Rightarrow C_4 (e^{-\pi/2} - e^{+\pi/2}) = 0$

$\Rightarrow \boxed{C_4 = 0} \quad \because e^{-\pi/2} - e^{\pi/2} \neq 0.$

$\Rightarrow \boxed{C_3 = 0}$

⑦ $\Rightarrow \boxed{C_1 = 0}$

⑧ $\Rightarrow \boxed{C_2 = 1}$

Thus ⑤ $\Rightarrow y = \sin x$

and ⑥ $\Rightarrow z = -\sin x$.

Example 2.

Show that the Euler-Lagrange equations for the functional

$$I = \int_a^b F(x, y, y', z, z')\, dx$$

are

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$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

and $\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0$

which admit the following 1st integrals

(i) $\frac{\partial F}{\partial y'} = c$ if F does not contain y.

(ii) $F - y'\frac{\partial F}{\partial y'} - z'\frac{\partial F}{\partial z'} = c$ if F does not contain x.

Solution.

Given that

$$I = \int_a^b F(x, y, y', z, z')\,dx.$$

Therefore,

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial z}\delta z + \frac{\partial F}{\partial z'}\delta z'\right] dx \quad \text{——— (1)}$$

having end point conditions

$$\delta y(a) = \delta y(b) = 0 \quad \text{——— (2)}$$

and $\delta z(a) = \delta z(b) = 0$.

Consider

$$\int_a^b \frac{\partial F}{\partial y'}\delta y'\,dx = \frac{\partial F}{\partial y'}\delta y\Big|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\,dx$$

$$= -\int_a^b \frac{d}{dx}\left(\partial F/\partial y'\right)\delta y\,dx.$$

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Similarly, $\int_a^b \frac{\partial F}{\partial z'} \delta z' dx = -\int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) \delta z \, dx$

Now equation (2) becomes

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial y}\delta y - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y + \frac{\partial F}{\partial z}\delta z - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right)\delta z \right] dx$$

For extremal $\delta I = 0$, therefore,

$$\int_a^b \left[\frac{\partial F}{\partial y}\delta y - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y + \frac{\partial F}{\partial z}\delta z - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right)\delta z \right] dx = 0$$

$$\Rightarrow \int_a^b \left[\left\{ \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right\}\delta y + \left\{ \frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) \right\} \right] dx = 0$$

$$\Rightarrow \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right]\delta y \, dx + \int_a^b \left[\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) \right]\delta z \, dx = 0$$

Since δy and δz are arbitrary, so $\delta y \neq 0$ and $\delta z \neq 0$, and therefore we must have

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

and $$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0.$$

(i)

The corresponding Euler Lagrange equation is

$$\partial F/\partial y - \frac{d}{dx}\left(\partial F/\partial y'\right) = 0 \qquad \text{——— (3)}$$

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If F does not contain y, then $\partial F/\partial y = 0$

So ③ $\Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$\Rightarrow \frac{\partial F}{\partial y'} = C$ (Constant).

(ii)

If F does not contain x, then

$$dF = \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz + \frac{\partial F}{\partial y'}dy' + \frac{\partial F}{\partial z'}dz' \quad \text{——— (4)}$$

③ $\Rightarrow \frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$

$= \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx}$ by chain rule.

$= y'\frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)$.

$\Rightarrow \frac{\partial F}{\partial y}dy = y'd\left(\frac{\partial F}{\partial y'}\right)$

Similarly,

$$\frac{\partial F}{\partial z}dz = z'd\left(\partial F/\partial z'\right)$$

Thus equation (4) becomes

$$dF = y'd(\partial F/\partial y') + z'd(\partial F/\partial z') + \frac{\partial F}{\partial y'}dy' + \partial F/\partial z'\,dz'.$$

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$$= d\left(y' \frac{\partial F}{\partial y'}\right) + d\left(z' \partial F/\partial z'\right)$$

$$= d\left(y' \frac{\partial F}{\partial y'} + z' \frac{\partial F}{\partial z'}\right)$$

$$\Rightarrow dF = d\left(y' \frac{\partial F}{\partial y'} + z' \frac{\partial F}{\partial z'}\right)$$

$$\Rightarrow dF - d\left(y' \frac{\partial F}{\partial y'} + z' \frac{\partial F}{\partial z'}\right) = 0$$

$$\Rightarrow d\left(F - y' \frac{\partial F}{\partial y'} - z' \frac{\partial F}{\partial z'}\right) = 0$$

On integrating, we get

$$\Rightarrow F - y' \frac{\partial F}{\partial y'} - z' \frac{\partial F}{\partial z'} = C.$$

Euler-Lagrange Equation, with higher order derivatives.

Let $I = \int_{x_1}^{x_2} F(x, y, y', y'', \cdots, y^{(n)})\, dx$,

where $y = y(x)$ is a continuous function having continuous derivatives satisfying the following end point conditions

$$y(x_1) = y'(x_1) = y''(x_1) = \cdots = y^{(n-1)}(x_1) = \text{Constant}$$

and $y(x_1) = y'(x_2) = y''(x_2) = \cdots = y^{(n-1)}(x_2) = \text{Const}^{t}$

For the extremal the given integral must satisfy the following

$$\frac{\partial F}{\partial y} + (-1)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2 \frac{d^2}{dx^2}\left(\partial F/\partial y''\right) +$$

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$$(-1)^3 \frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right) + \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0.$$

Proof.

Given integral is

$$I = \int_{x_1}^{x_2} F(x, y, y', y'', \cdots, y^{(n)})\, dx$$

Therefore, the variation of I as δI is

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y''}\delta y'' + \cdots + \frac{\partial F}{\partial y^{(n)}}\delta y^{(n)} \right] dx \quad \text{---------- (1)}$$

Consider,

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y' = \frac{\partial F}{\partial y'}\delta y \Big|_{x_1}^{x_2} - \int \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\, dx.$$

Since $y(x_1)$ and $y(x_2)$ are constants,

$\delta y(x_1) = 0 = \delta y(x_2)$.

Therefore,

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y' = -\int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\, dx.$$

Similarly,

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\delta y'' = -\int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\delta y'\, dx.$$

Integrating by parts, we get

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\delta y'' = -\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\delta y \Big|_{x_1}^{x_2} + (-1)^2 \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)\delta y\, dx$$

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$$= (-1)^2 \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) \delta y \, dx$$

Similarly,

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'''} \delta y''' = (-1)^3 \int_{x_1}^{x_2} \frac{d^3}{dx^3}\left(\partial F/\partial y'''\right) \delta y \, dx$$

Continuing in this way, we get

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y^{(n)}} \delta y^{(n)} = (-1)^n \int_{x_1}^{x_2} \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) \delta y \, dx.$$

Subsituting these expressions in eq (1), we get

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \delta y + (-1) \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \delta y + (-1)^2 \frac{d^2}{dx^2} \frac{\partial F}{\partial y''} \delta y \right.$$
$$\left. + (-1)^3 \frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right) \delta y + \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) \delta y \right] dx$$

For the extremal $\delta I = 0$, therefore,

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} + (-1) \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2 \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + \right.$$
$$\left. (-1)^3 \frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right) + \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) \right] \delta y \, dx = 0$$

By the fundamental theorem of calculus of variations, we have

$$\left[\frac{\partial F}{\partial y} + (-1) \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2 \frac{d^2}{dx^2}\left(\partial F/\partial y''\right) \right.$$
$$\left. + (-1)^3 \frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right) + \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) \right] \delta y = 0$$

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Since $\delta y \neq 0$, therefore we must have,

$$\frac{\partial F}{\partial y} + (-1)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + (-1)^3\frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right) + \cdots + (-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0$$

which is the Eulel- Lagrange equation in generalized form.

Example.

Find the extremal of the functional

$$I = \int_0^{\pi/2} (y''^2 - y^2 + x^2)\, dx$$

Subject to the B.Cs.

$y(0) = 1,\ y(\pi/2) = 0,\ y'(0) = 0,\ y'(\pi/2) = 1$

Solution.

Here $F = y''^2 - y^2 + x^2$. For the extremal F must satisfy the Eulel Lagrange. equation

$$\frac{\partial F}{\partial y} + (-1)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)^{0} + (-1)^2\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

$$\Rightarrow -2y + \frac{d^2}{dx^2}(2y'') = 0$$

$$\Rightarrow -y + y^{(4)} = 0$$

The Charac. equation is

$$D^4 - 1 = 0$$

$$(D^2-1)(D^2+1) = 0 \quad \Rightarrow D = \pm 1, \pm i$$

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$\Rightarrow y(x) = C_1 e^{x} + C_2 e^{-x} + C_3 \cos x + C_4 \sin x$ ——— (1)

$y(0) = 1$

$\Rightarrow C_1 + C_2 + C_3 = 1$ ——— (2)

$y(\pi/2) = 0 \Rightarrow C_1 e^{\pi/2} + C_2 e^{-\pi/2} + C_4 = 0$ ——— (3)

Diff. (1), w.r.t. x, we get

$y'(x) = C_1 e^{x} - C_2 e^{-x} - C_3 \sin x + C_4 \cos x$

$y'(0) = 0 \Rightarrow C_1 - C_2 + C_4 = 0$ ——— (4)

$y'(\pi/2) = 1 \Rightarrow C_1 e^{\pi/2} - C_2 e^{-\pi/2} - C_3 = 1$ ——— (5)

(2) + (5) $\Rightarrow C_1 + C_1 e^{\pi/2} + C_2 - C_2 e^{-\pi/2} = 2$

$\Rightarrow C_1(1 + e^{\pi/2}) + C_2(1 - e^{-\pi/2}) = 2$ ——— (6)

(3) − (4) $\Rightarrow C_1 e^{\pi/2} - C_1 + C_2 e^{-\pi/2} + C_2 = 0$

$\Rightarrow C_1(e^{\pi/2} - 1) + C_2(e^{-\pi/2} + 1) = 0$ ——— (7)

(6) $\Rightarrow C_1(1 + e^{\pi/2}) + C_2(1 - e^{-\pi/2}) - 2 = 0$

(7) $\Rightarrow C_1(e^{\pi/2} - 1) + C_2(e^{-\pi/2} + 1) + 0 = 0$

Solving, we get

$$\frac{C_1}{0 + 2(e^{-\pi/2} + 1)} = \frac{-C_2}{+2(e^{\pi/2} - 1)} = \frac{1}{(1 + e^{\pi/2})(e^{-\pi/2} + 1) - (1 - e^{-\pi/2})(e^{\pi/2} - 1)}$$

$$\Rightarrow C_1 = \frac{2(e^{-\pi/2} + 1)}{e^{-\pi/2} + 1 + e^{0} + e^{\pi/2} - e^{\pi/2} + 1 + e^{0} - e^{-\pi/2}}$$

$$= \frac{2(e^{-\pi/2} + 1)}{4} \Rightarrow C_1 = \frac{1}{2}(e^{-\pi/2} + 1)$$

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$$C_2 = \frac{-2(e^{\pi/2}-1)}{4} = -\frac{1}{2}(e^{\pi/2}-1).$$

(2) $\Rightarrow C_3 = 1 - C_1 - C_2$

$$= 1 - \frac{1}{2}(e^{-\pi/2}+1) + \frac{1}{2}(e^{\pi/2}-1)$$

$$= 1 - \frac{1}{2}e^{-\pi/2} - \frac{1}{2} + \frac{1}{2}e^{\pi/2} - \frac{1}{2}$$

$$= \frac{1}{2}(e^{\pi/2} - e^{-\pi/2})$$

$$\Rightarrow C_3 = \sinh(\pi/2)$$

(4) $\Rightarrow C_4 = C_2 - C_1$

$$= -\frac{1}{2}(e^{\pi/2}-1) - \frac{1}{2}(e^{-\pi/2}+1)$$

$$= -\frac{1}{2}e^{\pi/2} + \frac{1}{2} - \frac{1}{2}e^{-\pi/2} - \frac{1}{2}$$

$$= -\frac{1}{2}(e^{\pi/2} + e^{\pi/2})$$

$$\Rightarrow C_4 = -\cosh(\pi/2).$$

Thus

(1) $\Rightarrow y(x) = \frac{1}{2}(e^{-\pi/2}+1)e^{x} - \frac{1}{2}(e^{\pi/2}-1)e^{-x}$

$$+ \sinh(\pi/2)\cos x - \cosh(\tfrac{\pi}{2})\sin x.$$

* Thank

24/01/19

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Exercise

Q1. Find the extremals for

$$I = \int_0^1 (yy' + y''^2)\,dx \quad , y(0)=0, y(1)=2 \; ; y'(0)=1; \; y'(1)=4.$$

Solution.

Here $F = yy' + y''^2$. For the extremal F must satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2 \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

$$\Rightarrow y' - \frac{d}{dx}(y) + \frac{d^2}{dx^2}(2y'') = 0$$

$$\Rightarrow y' - y' + \frac{d^2}{dx^2}(2y'') = 0$$

$$\Rightarrow 2y^{(4)} = 0 \quad \Rightarrow y^{(4)} = 0$$

On integrating, we have

$$y^{(3)} = C_1$$

$$\Rightarrow y'' = C_1 x + C_2$$

$$\Rightarrow y'(x) = C_1 \frac{x^2}{2} + C_2 x + C_3 \quad \text{——— (1*)}$$

$$\Rightarrow y(x) = C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4 \quad \text{——— (1)}$$

$y(0) = 0 \Rightarrow \boxed{C_4 = 0}$

$y(1) = 2 \Rightarrow \frac{C_1}{6} + \frac{C_2}{2} + C_3 = 2$

$$\Rightarrow \frac{C_1 + 3C_2 + 6C_3}{6} = 2$$

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$\Rightarrow C_1 + 3C_2 + 6C_3 = 12$ ——— (2)

$y'(0) = 1 \Rightarrow \boxed{C_3 = 1}$

$y'(1) = 4 \Rightarrow \frac{C_1}{2} + C_2 + 1 = 4$

$\Rightarrow \frac{C_1}{2} + C_2 = 3 \Rightarrow C_1 + 2C_2 = 6$ ——— (3)

(2) $\Rightarrow C_1 + 3C_2 + 6 = 12$

$\Rightarrow C_1 + 3C_2 = 6$ ——— (4)

(4) − (3) $\Rightarrow \boxed{C_2 = 0}$

(3) $\Rightarrow \boxed{C_1 = 6}$

Thus

(1) $\Rightarrow y(x) = x^3 + x$.

Q2,

Find extremals of the problem

$\int_0^1 (360x^2y - y''^2)\,dx$, $y(0) = 0$, $y(1) = 0$

$y'(0) = 1$, $y'(1) = 5/2$.

Solution.

Here $F = 360x^2y - y''^2$. For the extremal F must satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2 \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

$$\Rightarrow 360x^2 - \frac{d}{dx}(0) + \frac{d^2}{dx^2}(-2y'') = 0.$$

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$\Rightarrow 360x^2 - 2y^{(4)} = 0$

$\Rightarrow 2y^{(4)} = 360x^2 \Rightarrow y^{(4)} = 180x^2$

$\Rightarrow y^{(3)} = 108\frac{x^3}{3} + C_1$ on integration.

$\Rightarrow y'' = 60\frac{x^4}{4} + C_1x + C_2$

$\Rightarrow y' = 15\cdot\frac{x^5}{5} + C_1\frac{x^2}{2} + C_2x + C_3$

$\Rightarrow y(x) = 3\frac{x^6}{6} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4$

$\Rightarrow y(x) = \frac{x^6}{2} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4$ ——— (1)

Applying B.Cs, we get

$y(0) = 0 \Rightarrow \boxed{C_4 = 0}$

$y(1) = 0 \Rightarrow \frac{1}{2} + \frac{C_1}{6} + \frac{C_2}{2} + C_3 = 0$

$\Rightarrow \frac{3 + C_1 + 3C_2 + 6C_3}{6} = 0$

$\Rightarrow C_1 + 3C_2 + 6C_3 = -3$ ——— (2)

$y'(0) = 1 \Rightarrow \boxed{C_3 = 0}$

$y'(1) = 5/2 \Rightarrow 3 + \frac{C_1}{2} + C_2 + 1 = 5/2$

$\Rightarrow 4 + \frac{C_1}{2} + C_2 = 5/2$

$\Rightarrow \frac{8 + C_1 + 2C_2}{2} = 5/2$

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$\Rightarrow C_1 + 2C_2 = -3$ ——— (3)

(2) $\Rightarrow C_1 + 3C_2 + 6 = -3$

$\Rightarrow C_1 + 3C_2 = -9$ ——— (4)

(4) − (3) $\Rightarrow \boxed{C_2 = -6}$

(3) $\Rightarrow C_1 - 12 = -3$

$\Rightarrow C_1 = -3 + 12 = 9 \Rightarrow \boxed{C_1 = 9}$

Thus

(1) $\Rightarrow y(x) = \frac{x^6}{2} + \frac{3}{2}x^3 - 3x^2 + x.$

Q3:

Find the extremals of the problem $I = \int_1^2 (y'^2 + z'^2 + z^2)\,dx$. Subject to $y(1) = 1,\ y(2) = 2;\ z(1) = 0;\ z(2) = 1.$

Solution.

Here $F = y'^2 + z'^2 + z^2$. For the extremal F must satisfy the Euler-Lagrange equations

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{——— (1)}$$

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0 \quad \text{——— (2)}$$

(1) $\Rightarrow 0 - \frac{d}{dx}(2y') = 0$

$\Rightarrow y'' = 0.$

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On integrating, we get

$y' = c_1$

$\Rightarrow y(x) = c_1 x + c_2$ ——— (3)

$y(1) = 1 \Rightarrow c_1 + c_2 = 1$ ——— (4)

$y(2) = 2 \Rightarrow 2c_1 + c_2 = 2$ ——— (5)

(5) − (4) = $\boxed{c_1 = 1}$

(4) $\Rightarrow$ $\boxed{c_2 = 0}$

Thus

(3) $\Rightarrow$ $y(x) = x$.

(2) $\Rightarrow$ $2z - \frac{d}{dx}(2z') = 0$

$\Rightarrow z - z'' = 0 \Rightarrow z'' - z = 0$.

The charac. eq. is

$D^2 - 1 = 0 \Rightarrow D = \pm 1$.

$\Rightarrow z(x) = c_3 e^x + c_4 e^{-x}$ ——— (6)

$z(1) = 0 \Rightarrow c_3 e^1 + c_4 e^{-1} = 0$ ——— (7)

$z(2) = 1 \Rightarrow c_3 e^2 + c_4 e^{-2} = 1$ ——— (8)

(7) $\Rightarrow c_3 e^2 + c_4 = 0$ ——— (9)

$\Rightarrow c_4 = -c_3 e^2$

(8) $\Rightarrow c_3 e^2 - c_3 e^2 e^{-2} = 1$

$\Rightarrow c_3(e^2 - 1) = 1$

$\Rightarrow c_3 = \frac{1}{e^2 - 1}$

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$$⑨ \Rightarrow \frac{e^2}{e^2-1} + C_4 = 0$$

$$\Rightarrow C_4 = \frac{-e^2}{e^2-1}.$$

Thus ⑥ $\Rightarrow Z(x) = \frac{1}{e^2-1} e^x - \frac{e^2}{e^2-1} e^{-x}.$

$$= \frac{e^x - e^2 e^{-x}}{e^2-1}$$

$$= \frac{e^x - e^2 e^{-x}}{e^{+1}(e^1 - e^{-1})}$$

$$= \frac{e^{-1}(e^x - e^2 e^{-x})}{e^1 - e^{-1}}$$

$$= \frac{e^{x-1} - e^{-(x-1)}}{e^1 - e^{-1}}$$

$$= \frac{(e^{x-1} - e^{-(x-1)})/2}{(e^1 - e^{-1})/2}$$

$$\Rightarrow Z(x) = \frac{\sinh(x-1)}{\sinh(1)}.$$

Q4.

Find extremals of problem

$$\int_0^{\pi} (2yz - 2y^2 + y'^2 - z'^2)\, dx.$$

Subject to $y(0) = 0$, $y(\pi) = 1$

$Z(0) = 0$, $Z(\pi) = -1$

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Solution.

Here $F = 2yz - 2y^2 + y'^2 - z'^2$. For the extremal F must satisfy the Euler-Lagrange equations.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{——— (1)}$$

and $$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0 \quad \text{——— (2)}$$

(1) $\Rightarrow 2z - 4y - \frac{d}{dx}(2y') = 0$

$\Rightarrow z - 2y - y'' = 0 \quad \text{——— (3)}$

(2) $\Rightarrow 2y - \frac{d}{dx}(-2z') = 0$

$\Rightarrow y + z'' = 0 \quad \text{——— (4)}$

$\Rightarrow y = -z'' \Rightarrow y'' = -z^{(4)}$.

(3) $\Rightarrow z^{(4)} + 2z'' + z = 0$

The charac. equation, is

$$D^4 + 2D^2 + 1 = 0$$

$\Rightarrow (D^2+1)^2 = 0$

$\Rightarrow (D^2+1)(D^2+1) = 0$

$\Rightarrow D = \pm i, \pm i$

$\Rightarrow z(x) = (C_1 + C_2 x)\cos x + (C_3 + C_4 x)\sin x \quad \text{——— (5)}$

$z(0) = 0 \Rightarrow \boxed{C_1 = 0}$

$z(\pi) = -1 \Rightarrow (C_2 \pi)(-1) = -1$

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$\Rightarrow \boxed{C_2 = 1/\pi}$

$z'(x) = (C_1 + C_2 x)(-\sin x) + C_2 \cos x$
$+ (C_3 + C_4 x)\cos x + C_4 \sin x$
$= -C_1 \sin x - C_2 x \sin x + C_2 \cos x$
$+ C_3 \cos x + C_4 x \cos x + C_4 \sin x$

$z'' = -C_1 \cos x - C_2(\sin x + x\cos x)$
$+ C_2(-\sin x) + C_3(-\sin x) + C_4(\cos x$
$- x\sin x) + C_4 \cos x.$

$= C_1 \cos x + C_2(-x\cos x - 2\sin x)$
$- C_3 \sin x + C_4(2\cos x - x\sin x).$

④ $\Rightarrow$
$y = -C_1 \cos x + C_2(x\cos x + 2\sin x)$
$+ C_3 \sin x + C_4(x\sin x - 2\cos x)$ ——⑥

$y(0) = 0 \Rightarrow -C_1 - 2C_4 = 0$
$\Rightarrow -2C_4 = 0 \Rightarrow \boxed{C_4 = 0}$ $\because \boxed{C_4 = 0}$

$y(\pi) = 1$
$\Rightarrow C_2(-\pi) = 1$
$\Rightarrow \boxed{C_2 = -1/\pi}$

Thus
⑥ $\Rightarrow y(x) = -\frac{1}{\pi}(x\cos x + 2\sin x) + C_3 \sin x$

$z(x) = -\frac{x}{\pi}\cos x + C_3 \sin x$; C_3 is arbitrary.

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Q5. Find the extremals for the problem

$$I = \int_{-1}^{0} (240y - y'''^2)\, dx.$$

$y(-1)=1$, $y(0)=0$, $y'(-1) = -9/2$

$y'(0)=0$, $y''(-1)=16$, $y''(0)=0$.

Solution.

Here $F = 240y - (y''')^2$. For the extremal F must satisfy the Euler-Lagrange equation.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)^{0} + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)^{0} - \frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right) = 0$$

$$\Rightarrow 240 - \frac{d^3}{dx^3}(-2y''') = 0$$

$$\Rightarrow 120 + y^{(6)} = 0$$

$$\Rightarrow y^{(6)} = -120$$

On integrating, we get

$$y^{(5)} = -120x + C_1$$

$$y^{(4)} = -120\frac{x^2}{2} + C_1 x + C_2$$

$$y^{(3)} = -60\frac{x^3}{3} + C_1\frac{x^2}{2} + C_2 x + C_3$$

$$y'' = -20\frac{x^4}{4} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4$$

$$y' = -5\frac{x^5}{5} + C_1\frac{x^4}{24} + C_2\frac{x^3}{6} + C_3\frac{x^2}{2} + C_4 x + C_5$$

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$$y(x) = -\frac{x^6}{6} + C_1\frac{x^5}{120} + C_2\frac{x^4}{24} + C_3\frac{x^3}{6} + C_4\frac{x^2}{2} + C_5 x + C_6 \quad \text{---------- (1)}$$

$y(0)=0 \Rightarrow$ from (1) $\boxed{C_6 = 0}$

$y'(0)=0 \Rightarrow \boxed{C_5 = 0}$

$y''(0)=0 \Rightarrow \boxed{C_4 = 0}$

$y(-1)=1$

$$\Rightarrow -\frac{1}{6} - \frac{C_1}{120} + \frac{C_2}{24} - \frac{C_3}{6} + \frac{C_4}{2} - C_5 + C_6 = 1$$

$$\Rightarrow -\frac{1}{6} - \frac{C_1}{120} + \frac{C_2}{24} - \frac{C_3}{6} = 1$$

$$\Rightarrow \frac{-20 - C_1 + 5C_2 - 20C_3}{120} = 1$$

$$\Rightarrow -C_1 + 5C_2 - 20C_3 = 120 + 20$$

$$\Rightarrow C_1 + 5C_2 + 20C_3 = -140 \quad \text{---------- (2)}$$

$y'(-1) = -9/2$

$$\Rightarrow 1 + \frac{C_1}{24} + \frac{C_2}{6} + \frac{C_3}{2} - C_4 + C_5 = -\frac{9}{2}$$

$$\Rightarrow 1 + \frac{C_1}{24} - \frac{C_2}{6} + \frac{C_3}{2} = -\frac{9}{2}$$

$$\Rightarrow \frac{24 + C_1 - 4C_2 + 12C_3}{24} = -9/2$$

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$24 + C_1 - 4C_2 + 12C_3 = -108$

$\Rightarrow C_1 - 4C_2 + 12C_3 = -132$ ——— (3)

$y''(-1) = 16$

$\Rightarrow -5 - \frac{C_1}{6} + \frac{C_2}{2} - C_3 + C_4 = 16$

$\Rightarrow -5 - \frac{C_1}{6} + \frac{C_2}{2} - C_3 = 16$

$\Rightarrow -\frac{C_1}{6} + \frac{C_2}{2} - C_3 = 21$

$\Rightarrow \frac{-C_1 + 3C_2 - 6C_3}{6} = 21 \Rightarrow -C_1 + 3C_2 - 6C_3 = 126$

$\Rightarrow C_1 - 3C_2 + 6C_3 = -126$ ——— (4)

(2) − (3) $\Rightarrow -5C_2 + 4C_2 + 20C_3 - 12C_3 = -140 + 132$

$\Rightarrow -C_2 + 8C_3 = -8$ ——— (5)

(3) − (4) $\Rightarrow -4C_2 + 12C_3 + 3C_2 - 6C_3 = -132 + 126$

$\Rightarrow -C_2 + 6C_3 = -6$

$\Rightarrow C_2 - 6C_3 = 6$ ——— (6)

(5) + (6) $\Rightarrow 2C_3 = -2$

$\Rightarrow \boxed{C_3 = -1}$

(6) $\Rightarrow C_2 + 6 = 6 \Rightarrow \boxed{C_2 = 0}$

(2) $\Rightarrow C_1 - 20 = -140$

$\Rightarrow \boxed{C_1 = -120}$

Thus (1) $\Rightarrow y = -\frac{x^6}{6} - x^5 - \frac{x^3}{6}$ Ans

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Q6:-

Find the extremals of the problem

$$I = \int_0^1 (y'')^2 \, dx \; ; \; y(0) = 0, \; y(1) = 1/2$$
$$y'(0) = 0, \quad y'(1) = 1.$$

Solution.

Here $F = (y'')^2$. For the extremal F must satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

$$\Rightarrow \frac{d^2}{dx^2}(2y'') = 0$$

$$\Rightarrow y^{(4)} = 0$$

Integrating, we get

$$y^{(3)} = C_1$$
$$y'' = C_1 x + C_2$$
$$y' = C_1 \frac{x^2}{2} + C_2 x + C_3$$
$$y(x) = C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4 \quad \text{——} (1)$$

$y(0) = 0 \Rightarrow \boxed{C_4 = 0}$

$y(1) = 1/2 \Rightarrow \frac{C_1}{6} + \frac{C_2}{2} + C_3 = 1/2 \quad \text{——} (2)$

$y'(0) = 0 \Rightarrow \boxed{C_3 = 0}$

$y'(1) = 1 \Rightarrow \frac{C_1}{2} + C_2 + C_3 = 1$

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$\Rightarrow \frac{C_1}{2} + C_2 = 1 \Rightarrow C_1 + 2C_2 = 2$ ——— (3)

(2) $\Rightarrow \frac{C_1}{6} + \frac{C_2}{2} = \frac{1}{2} \Rightarrow \frac{C_1 + 3C_2}{6} = \frac{1}{2}$

$\Rightarrow C_1 + 3C_2 = 3$ ——— (4)

(4) − (3) $\Rightarrow$ $\boxed{C_2 = 1}$

(3) $\Rightarrow$ $\boxed{C_1 = 0}$

Thus (1) $\Rightarrow$ $y(x) = \frac{x^2}{2}$

Q7:

Find the extremals of the problem

$I = \int_{-1}^{1} (2xy - y'^2 + z'^2/3)\,dx$, $y(1) = 0$, $y(-1) = 2$

$z(1) = 1$, $z(-1) = -1$.

Solution.

Here $F = 2xy - y'^2 + z'^2/3$. For the extremal F must satisfy the Euler Lagrange equations.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{——— (1)}$$

and $$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0 \quad \text{——— (2)}$$

(1) $\Rightarrow 2x - \frac{d}{dx}(-2y') = 0$

$\Rightarrow x + y'' = 0$ ——— (3)

(2) $\Rightarrow 0 - \frac{d}{dx}\left(\frac{2}{3}z'\right) = 0$

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$\Rightarrow z'' = 0$

On integration, we get

$z' = C_1$

$\Rightarrow z(x) = C_1 x + C_2$

$z(1) = 1 \Rightarrow C_1 + C_2 = 1$ ——— (4)

$z(-1) = -1 \Rightarrow -C_1 + C_2 = -1$ ——— (5)

(4) + (5) $\Rightarrow 2C_2 = 0 \Rightarrow \boxed{C_2 = 0}$

(4) $\Rightarrow \boxed{C_1 = 1}$

Thus $z(x) = x$.

(3) $\Rightarrow y'' = -x$

Integrating, we get

$y' = -\frac{x^2}{2} + C_3$

$y(x) = -\frac{x^3}{6} + C_3 x + C_4$

$y(1) = 0 \Rightarrow -\frac{1}{6} + C_3 + C_4 = 0$

$\Rightarrow C_3 + C_4 = 1/6$ ——— (6)

$y(-1) = 2 \Rightarrow \frac{1}{6} - C_3 + C_4 = 2$

$\Rightarrow -C_3 + C_4 = 2 - \frac{1}{6}$

$\Rightarrow -C_3 + C_4 = 11/6$ ——— (7)

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⑥ + ⑦ $\Rightarrow 2C_4 = \frac{11}{6} + \frac{1}{6} = \frac{12}{6} = 2$

$\Rightarrow \boxed{C_4 = 1}$

⑥ $C_3 + 1 = \frac{1}{6} \Rightarrow C_3 = \frac{1}{6} - 1 = -5/6$

$\Rightarrow \boxed{C_3 = -5/6}$

Thus

$$y(x) = -\frac{x^3}{6} - \frac{5}{6}x + 1 = \frac{-1}{6}(x^3 + 5x - 6).$$

Q8:-

Find the extremals for the problem

$$I = \int_0^1 (y'^2 + z'^2 + 2y)\,dx.$$

$y(0) = 1$, $y(1) = 3/2$; $z(0) = 0$, $z(1) = 1$.

Solution.

Here $F = y'^2 + z'^2 + 2y$. For the extremals F must satisfy the Euler-Lagrange equations.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{——— (1)}$$

and $$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0 \quad \text{——— (2)}$$

(1) $\Rightarrow 2 - \frac{d}{dx}(2y') = 0$

$\Rightarrow 1 - y'' = 0$

$\Rightarrow y'' = 1$

Integrating, we get

$y' = x + C_1$

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$$y(x) = \frac{x^2}{2} + C_1 x + C_2 \quad \text{——— (3)}$$

$y(0) = 1 \Rightarrow \boxed{C_2 = 0}$

$y(1) = 3/2 \Rightarrow \frac{1}{2} + C_1 = 3/2$

$\Rightarrow C_1 = 3/2 - \frac{1}{2} = 1$

$\Rightarrow \boxed{C_1 = 1}$

Thus

(3) $\Rightarrow y(x) = \frac{x^2}{2} + x$

(2) $\Rightarrow 0 - \frac{d}{dx}(xz') = 0$

$\Rightarrow z'' = 0$

Integrating, we get

$z' = C_3$

$\Rightarrow z(x) = C_3 x + C_4 \quad \text{——— (4)}$

$z(0) = 0 \Rightarrow \boxed{C_4 = 0}$

$z(1) = 1 \Rightarrow \boxed{C_3 = 1}$

Thus (4) $\Rightarrow z(x) = x.$

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Eules-Lagrange Equation for two independent variables.

Let $F = F(x, y, u, u_x, u_y)$ where x and y are independent variables. In this case, we have to extremize the functional

$$I = \iint_R F(x, y, u, u_x, u_y)\, dx\, dy$$

where u has different values inside the region R but is prescribed on the boundary of the region R.

Moreover u is supposed to possess continuous partial derivatives upto second order, is given by

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0.$$

Proof.

Given that

$$I = \iint_R F(x, y, u, u_x, u_y)\, dx\, dy \quad \text{——— (1)}$$

The variation δI in I is given by

$$\delta I = \iint_R \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y\right) dx\, dy \quad \text{——— (2)}$$

Note that

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u + \frac{\partial F}{\partial u_x}\frac{\partial}{\partial x}\delta u$$

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$$\Rightarrow \frac{\partial}{\partial u_x}\delta u_x = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u \quad \text{---} (3)$$

Similarly,

$$\frac{\partial}{\partial u_y}\delta u_y = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u \quad \text{---} (4)$$

Subsituting (3) and (4) in (2), we get

$$\delta I = \iint_R \left[\frac{\partial F}{\partial u}\delta u + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u\right] dx\,dy.$$

$$= \iint_R \left[\frac{\partial F}{\partial u}\delta u - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right)\right] dx\,dy$$

$$= \iint_R \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right]\delta u\, dx\,dy + \iint_R \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right)\right] dx\,dy$$

$$\Rightarrow \delta I = I_1 + I_2 \quad \text{---} (5)$$

Consides

$$I_2 = \iint_R \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right)\right] dx\,dy$$

As Green's theorem is

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy)$$

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So $I_2 = \oint_C \left(-\frac{\partial F}{\partial u_y} \delta u \, dx + \frac{\partial F}{\partial u_x} \delta u \, dy \right)$

$= \oint \left(\frac{\partial F}{\partial u_x} dy - \frac{\partial F}{\partial u_y} dx \right) \delta u.$

Since u is prescribed (fixed) on the boundary, $\delta u = 0$.

$\Rightarrow I_2 = 0$

Now equation (5) becomes

$$\delta I = \iint_R \left(-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) + \frac{\partial F}{\partial u} - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) \right) \delta u \, dx \, dy.$$

For extremal $\delta I = 0$, therefore

$$\iint_R \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) \right] \delta u \, dx \, dy = 0$$

Since δu is arbitrary, by generalized form of fundamental theorem of calculus of variations, we have

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0.$$

which is required Euler-Lagrange equation.

Constrained Extrema.

The variational problems in which we have to find stationary values of the functional of the form

$$I = \int_{x_1}^{x_2} F(x, y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n') \, dx$$

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where the arguments of F are subject to some additional conditions or constraints such as

(i) $G(x, y_1, y_2, \cdots, y_n) = \text{Constant}$

or (ii) $G(x, y_1, y_2, \cdots y_n, y_1', y_2', \cdots y_n') = \text{Constant}$

or (iii) $\int_{x_1}^{x_2} G(x, y_1, y_2, \cdots y_n, y_1', y_2', \cdots, y_n')dx = \text{Constant}$

are called variational problems with constraints or variational problems with side conditions or isoperimetrical problems.

The Euler - Lagrange equation in this case will be

$$\frac{\partial H}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial H}{\partial y_i'}\right) = 0 \quad ; i = 1, 2, \cdots n.$$

where $H = F + \lambda G$.

Formulate and solve the Dido's problem.

2017/A Find the closed curve of given length which encloses maximum area.

Formulation.

Since the area enclosed by the curve $y = y(x)$ between the lines $x = a$ and $x = b$ and the x-axis is given by

$$\text{Area} = \int_a^b y(x)\,dx \qquad \text{---------- (1)}$$

and the length l of the same curve

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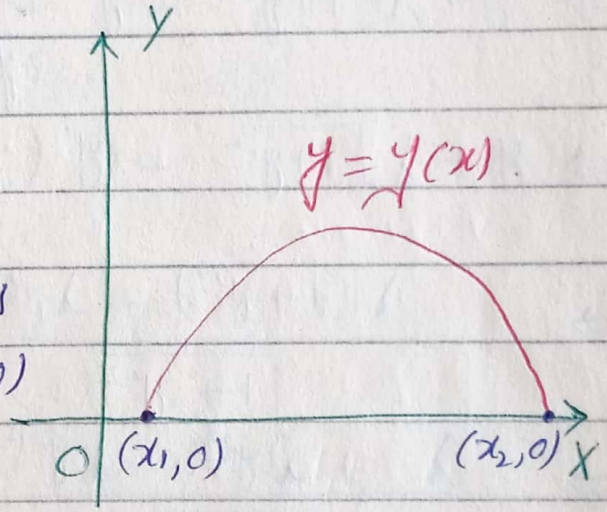
between $x=a$ and $x=b$ is given by

$$l = \int_a^b ds = \int_a^b \sqrt{1+y'^2}\, dx \qquad \text{——— (2)}$$

Thus the problem reduces to that of maximizing the area in eq (1) subject to the condition given in (2).

Solution.

Let us assume that $y = y(x)$ is the curve which meets the x-axis at points $(x_1, 0)$ and $(x_2, 0)$ and encloses maximum area.



Since area enclosed $= \int_{x_1}^{x_2} y(x)\, dx$, therefore we have to extremize the functional

$$I = \int_{x_1}^{x_2} y\, dx$$

Subject to the condition

$$\int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx = l, \quad (l > x_2 - x_1)$$

Here $F = y$ and $G = \sqrt{1+y'^2}$

and therefore,

$$H = F + \lambda G$$
$$= y + \lambda \sqrt{1+y'^2}$$

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Since there is no explicit dependence on x, therefore the corresponding Euler-Lagrange equation is

$$H = y' \frac{\partial H}{\partial y'} = c_1 \quad \text{(a constant)}$$

$$\Rightarrow y + \lambda\sqrt{1+y'^2} - y' \frac{\partial}{\partial y'}\left(y + \lambda\sqrt{1+y'^2}\right) = c_1$$

$$\Rightarrow y + \lambda\sqrt{1+y'^2} - y'\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = c_1$$

$$\Rightarrow \frac{\lambda(1+y'^2) - \lambda y'^2}{\sqrt{1+y'^2}} = c_1 - y$$

$$\Rightarrow \frac{\lambda + \lambda y'^2 - \lambda y'^2}{\sqrt{1+y'^2}} = c_1 - y$$

$$\Rightarrow \frac{\lambda}{\sqrt{1+y'^2}} = c_1 - y$$

Squaring on both sides, we get

$$\frac{\lambda^2}{1+y'^2} = (c_1 - y)^2$$

$$\Rightarrow 1 + y'^2 = \frac{\lambda^2}{(c_1 - y)^2}$$

$$\Rightarrow y'^2 = \frac{\lambda^2}{(c_1 - y)^2} - 1$$

$$\Rightarrow y'^2 = \frac{\lambda^2 - (c_1 - y)^2}{(c_1 - y)^2}$$

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$$\Rightarrow y' = \sqrt{\frac{\lambda^2 - (C_1 - y)^2}{(C_1 - y)^2}}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{\lambda^2 - (C_1 - y)^2}{(C_1 - y)^2}}$$

$$\Rightarrow \int \frac{C_1 - y}{\sqrt{\lambda^2 - (C_1 - y)^2}}\, dy = \int dx$$

Separating variables.

Put $C_1 - y = u \Rightarrow du = -dy$.

$$\Rightarrow \int \frac{-u\,du}{\sqrt{\lambda^2 - u^2}} = x + C_2$$

$$\Rightarrow \int (\lambda^2 - u^2)^{-1/2} (-u)\, du = x + C_2$$

$$\Rightarrow \frac{1}{2} \int (\lambda^2 - u^2)^{-\frac{1}{2}} (-2u\, du) = x + C_2$$

$$\Rightarrow \frac{1}{2} \frac{(\lambda^2 - u^2)^{1/2}}{1/2} = x + C_2$$

$$\Rightarrow (\lambda^2 - u^2)^{1/2} = x + C_2$$

$$\Rightarrow \lambda^2 - u^2 = (x + C_2)^2$$

$$\Rightarrow \lambda^2 - (C_1 - y)^2 = (x + C_2)^2$$

$$\Rightarrow \lambda^2 = (x + C_2)^2 + (C_1 - y)^2$$

$$\Rightarrow (x + C_2)^2 + (y - C_1)^2 = \lambda^2$$

$\because (y - C_1)^2 = (C_1 - y)^2$

which is the equation of a circles at having centre at $(-C_2, C_1)$ and having radius λ.

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Centroid.

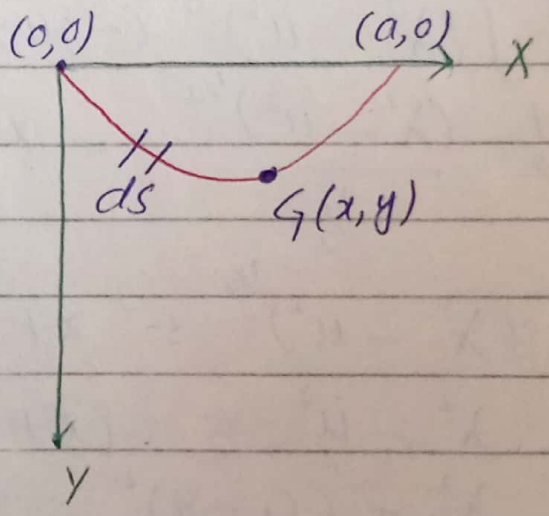
The point of a body where whole mass of the body acts is called centroid.

Example 1. 2018/S.

A uniform cable is fixed at its ends at the same level in space and is allowed to hang under gravity. Find the final shape of the cable.

Solution.

The final shape of the cable will correspond to the state of stable equilibrium.



Let $(0,0)$ and $(a,0)$ be the positions of the endpoints of the cable. The P.E. of the cable is given by $V = mg\bar{y}$, where $\bar{y}$ is the y-coordinate of the centroid of the cable. The minimum value of V corresponds to the minimum value of $\bar{y}$.

Now the y-coordinate of the centroid of the curve $y = y(x)$ is given by

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$$\bar{y} = \frac{\int_0^a \rho y\, ds}{\int_0^a \rho\, ds} = \frac{\rho \int_0^a y\, ds}{\rho \int_0^a ds} = \frac{\int_0^a y\sqrt{1+y'^2}\, dx}{l}$$

$$\Rightarrow \bar{y} = \frac{1}{l}\int_0^a y\sqrt{1+y'^2}\, dx$$

where l is the length of the cable and ρ its mass per unit length. Thus, we have to find the curve $y = y(x)$ which minimizes $\bar{y}$ subject to the condition

$$\int_0^a \sqrt{1+y'^2}\, dx = l.$$

Here $F = y\sqrt{1+y'^2}$; $G = \sqrt{1+y'^2}$

so $H = F + \lambda G = y\sqrt{1+y'^2} + \lambda\sqrt{1+y'^2}$

Since H is not explicit function of x, so the corresponding Eulers-Lagrange equation is

$$H - y'\left(\frac{\partial H}{\partial y'}\right) = C_1 \quad \text{(Constant)}$$

$$\Rightarrow y + \sqrt{1+y'^2} + \lambda\sqrt{1+y'^2} - y'\left(\frac{(y+\lambda)y'}{\sqrt{1+y'^2}}\right) = C_1$$

$$\Rightarrow \frac{(y+\lambda)(1+y'^2) - (y+\lambda)y'^2}{\sqrt{1+y'^2}} = C_1$$

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$$\Rightarrow \frac{y(1+y'^2) + \lambda(1+y'^2) - (y+\lambda)y'^2}{\sqrt{1+y'^2}} = c_1$$

$$\Rightarrow \frac{(y+\lambda) + (y+\lambda)y'^2 - (y+\lambda)y'^2}{\sqrt{1+y'^2}} = c_1$$

$$\Rightarrow \frac{y+\lambda}{\sqrt{1+y'^2}} = c_1$$

On squaring, we have

$$(y+\lambda)^2 = c_1^2(1+y'^2)$$

$$\Rightarrow 1+y'^2 = \frac{(y+\lambda)^2}{c_1^2}$$

$$\Rightarrow y'^2 = \frac{(y+\lambda)^2}{c_1^2} - 1$$

$$\Rightarrow y'^2 = \frac{(y+\lambda)^2 - c_1^2}{c_1^2}$$

$$\Rightarrow y' = \sqrt{\frac{(y+\lambda)^2 - c_1^2}{c_1^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{(y+\lambda)^2 - c_1^2}}{c_1}$$

$$\Rightarrow \int \frac{c_1}{\sqrt{(y+\lambda)^2 - c_1^2}}\, dy = \int dx$$

separating variables

$$\Rightarrow c_1 \int \frac{dy}{\sqrt{(y+\lambda)^2 - c_1^2}} = x + c_2$$

$$\Rightarrow c_1 \cosh^{-1}\left(\frac{y+\lambda}{c_1}\right) = x + c_2 \quad \text{---} \quad (A)$$

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Applying the conditions

$y(0) = 0$

$\Rightarrow c_1 \cosh^{-1}(\lambda/c_1) = c_2$

$\Rightarrow \cosh^{-1}(\lambda/c_1) = c_2/c_1$

$\Rightarrow \frac{\lambda}{c_1} = \cosh(c_2/c_1)$ ——— (1)

and $y(a) = 0$

$\Rightarrow c_1 \cosh^{-1}(\lambda/c_1) = a + c_2$

$\Rightarrow \cosh^{-1}(\lambda/c_1) = \frac{a+c_2}{c_1}$

$\Rightarrow \lambda/c_1 = \cosh\left(\frac{a+c_2}{c_1}\right)$ ——— (2)

From (1) and (2), we have

$\cosh(c_2/c_1) = \cosh\left(\frac{a+c_2}{c_1}\right)$

$\Rightarrow \frac{e^{c_2/c_1} + e^{-c_2/c_1}}{2} = \frac{e^{\frac{(a+c_2)}{c_1}} + e^{-\frac{(a+c_2)}{c_1}}}{2}$

$\Rightarrow e^{c_2/c_1} + e^{-c_2/c_1} - e^{a/c_1} \cdot e^{c_2/c_1} - e^{-a/c_1} \cdot e^{-c_2/c_1} = 0$

$\Rightarrow e^{c_2/c_1}(1 - e^{a/c_1}) + e^{-c_2/c_1}(1 - e^{-a/c_1}) = 0$

$\Rightarrow e^{c_2/c_1}(1 - e^{a/c_1}) + \frac{1}{e^{c_2/c_1}}\left(1 - \frac{1}{e^{a/c_1}}\right) = 0$

$\Rightarrow \frac{e^{a/c_1}(e^{c_2/c_1})^2(1 - e^{a/c_1}) + (e^{a/c_1} - 1)}{e^{c_2/c_1} e^{a/c_1}} = 0$

$\Rightarrow e^{a/c_1}(e^{c_2/c_1})^2(1 - e^{a/c_1}) - (1 - e^{a/c_1}) = 0$

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$$\Rightarrow (1-e^{a/c_1})(e^{a/c_1}(e^{c_2/c_1})^2 - 1) = 0$$

$$\Rightarrow e^{a/c_1}(e^{c_2/c_1})^2 - 1 = 0 \quad \because 1-e^{a/c_1} \neq 0.$$

$$\Rightarrow e^{a/c_1}(e^{c_2/c_1})^2 = 1$$

$$\Rightarrow e^{a/c_1} e^{2c_2/c_1} = e^0$$

$$\Rightarrow e^{\frac{a}{c_1}+2\frac{c_2}{c_1}} = e^0$$

$$\Rightarrow \frac{a}{c_1} + \frac{2c_2}{c_1} = 0$$

$$\Rightarrow \frac{a+2c_2}{c_1} = 0$$

$$\Rightarrow a+2c_2 = 0$$

$$\Rightarrow 2c_2 = -a \Rightarrow \boxed{c_2 = -a/2}$$

Thus equation (A) becomes

$$c_1 \cosh^{-1}\left(\frac{y+\lambda}{c_1}\right) = x - \frac{a}{2}$$

$$\Rightarrow \cosh^{-1}\left(\frac{y+\lambda}{c_1}\right) = \frac{x}{c_1} - \frac{a}{2c_1}$$

$$\Rightarrow \frac{y+\lambda}{c_1} = \cosh\left(\frac{x}{c_1} - \frac{a}{2c_1}\right)$$

$$\Rightarrow \frac{y}{c_1} + \lambda/c_1 = \cosh\left(x/c_1 - a/2c_1\right)$$

using ①

$$\Rightarrow y/c_1 + \cosh\left(-\frac{a}{2c_1}\right) = \cosh\left(x/c_1 - a/2c_1\right)$$

$$\Rightarrow y = c_1\left[\cosh\left(x/c_1 - a/2c_1\right) - \cosh\left(a/2c_1\right)\right]$$

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This curve is called catenary.

Special

Example 2. 2010, 2011, 2015

Show that a solid of revolution which for a given surface area has maximum volume is a sphere.

OR Find the curve which generates a surface of revolution of a given area which encloses maximum volume.

Solution.

Let a curve $y = y(x)$ with $y(0) = 0, y(a) = 0$ be rotated about X-axis so as to generate a surface of revolution.

An element of surface area is $2\pi y\,ds$.

Therefore, the total surface area A will be

$$A = 2\pi \int_0^a y\,ds = 2\pi \int_0^a y\sqrt{1+y'^2}\,dx.$$

The volume element of solid of revolution is $\pi y^2 dx$.

Therefore, the total volume V is

$$V = \pi \int_0^a y^2\,dx$$

Thus, we have to find the curve $y = y(x)$ with end point conditions $y(0) = 0, y(a) = 0$ which maximize the integral

$$I = \pi \int_0^a y^2\,dx$$

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Subject to the condition

$$A = 2\pi \int_0^a y\sqrt{1+y'^2}\,dx.$$

Here $F = y^2$, $G = y\sqrt{1+y'^2}$

So $H = F + \lambda G$

$$= y^2 + \lambda y\sqrt{1+y'^2}.$$

Since H does not depend on x explicitly, therefore the Euler-Lagrange equation in this case is

$$H - y'\left(\frac{\partial H}{\partial y'}\right) = C \quad \text{(a constant)}$$

$$\Rightarrow y^2 + \lambda y\sqrt{1+y'^2} - y'\left(\frac{\lambda y y'}{\sqrt{1+y'^2}}\right) = C$$

$$\Rightarrow \lambda y\sqrt{1+y'^2} - \frac{\lambda y y'^2}{\sqrt{1+y'^2}} = C - y^2$$

$$\Rightarrow \frac{\lambda y(1+y'^2) - \lambda y y'^2}{\sqrt{1+y'^2}} = C - y^2$$

$$\Rightarrow \frac{\lambda y + \lambda y y'^2 - \lambda y y'^2}{\sqrt{1+y'^2}} = C - y^2$$

$$\Rightarrow \frac{\lambda y}{\sqrt{1+y'^2}} = C - y^2$$

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$\Rightarrow \lambda y = \sqrt{1+y'^2}\ (c - y^2)$ ——— (1)

Since y, y' are continuous on the interval $(0, a)$. Therefore, y' will be defined at any point in $(0, a)$. Then y'^2 will be non negative and finite.

(1) $\Rightarrow \quad 0 = \sqrt{1+\alpha}\ (c - 0)$

where $y'^2 = \alpha$ any finite +ve number

$\Rightarrow \boxed{c = 0}$

Therefore, $\lambda y = \sqrt{1+y'^2}\ (-y^2)$

Squaring on both sides, we get

$$\lambda^2 y^2 = (1+y'^2)\, y^4$$

$$\Rightarrow \lambda^2 = (1+y'^2)\, y^2$$

$$\Rightarrow \lambda^2 = y^2 + y^2 y'^2$$

$$\Rightarrow \lambda^2 - y^2 = y^2 y'^2$$

$$\Rightarrow y'^2 = \frac{\lambda^2 - y^2}{y^2}$$

$$\Rightarrow y' = \sqrt{\frac{\lambda^2 - y^2}{y^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{\lambda^2 - y^2}}{y}$$

$$\Rightarrow \int \frac{y\, dy}{\sqrt{\lambda^2 - y^2}} = \int dx$$

Separating variables

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$$\Rightarrow \int (\lambda^2 - y^2)^{-1/2} y\,dy = x + c_1$$

$$\Rightarrow -\frac{1}{2}\int (\lambda^2 - y^2)^{-1/2}(-2y\,dy) = x + c_1$$

$$\Rightarrow -\frac{1}{2}\frac{(\lambda^2 - y^2)^{1/2}}{1/2} = x + c_1$$

$$\Rightarrow -\sqrt{\lambda^2 - y^2} = x + c_1$$

Squaring on both sides, we get

$$\lambda^2 - y^2 = (x + c_1)^2$$

$$\Rightarrow (x + c_1)^2 + y^2 = \lambda^2.$$

which is equation of circle with centre at $(-c_1, 0)$ and radius λ. This shows that the curve that generates a surface of revolution of given surface area but of maximum volume is a circle. Hence the surface of revolution will be a sphere.

Example 3. 2018/S.

Find the eigenvalues and eigenfunctions of the functional

$$I = \int_0^3 [(2x+3)^2 y'^2 - y^2]\,dx$$

subject to the end point conditions $y(0) = 0$, $y(3) = 0$ and the side condition $\int_0^3 y^2\,dx = 1$

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Solution.

Here $F = (2x+3)^2 y'^2 - y^2$, $G = y^2$.

So $H = F + \lambda G$

$= (2x+3)^2 y'^2 - y^2 + \lambda y^2$.

The corresponding Euler-Lagrange equation is

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$

$$\Rightarrow -2y + 2\lambda y - \frac{d}{dx}\left[2y'(2x+3)^2\right] = 0$$

$$\Rightarrow \frac{d}{dx}\left[(2x+3)^2 y'\right] + y - \lambda y = 0.$$

$$\Rightarrow (2x+3)^2 y'' + 2(2x+3)(2)y' + (1-\lambda)y = 0$$

$$\Rightarrow (2x+3)^2 y'' + 4(2x+3)y' + (1-\lambda)y = 0 \quad \text{---} (1)$$

Which is an Cauchy - Euler equation.

Put $2x+3 = e^t$; $\frac{dx}{dt} = e^t/2$.

$\Rightarrow \ln(2x+3) = t$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$ by chain rule.

$$\Rightarrow y' = 2e^{-t}\frac{dy}{dt}.$$

Again using chain rule of derivatives

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right) dt/dx$$

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$$\Rightarrow y'' = \frac{d}{dt}\left(2e^{-t}\frac{dy}{dt}\right)(2e^{-t})$$

$$= 2e^{-t}\frac{d}{dt}\left(2e^{-t}\frac{dy}{dt}\right)$$

$$= 4e^{-t}\left(e^{-t}\frac{d^2y}{dt^2} + \frac{dy}{dt}(e^{-t})\right)$$

$$\Rightarrow y'' = 4e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Now equation ① becomes

$$e^{2t}\cdot 4e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 4e^{t}\cdot 2e^{-t}\frac{dy}{dt} + (1-\lambda)y = 0$$

$$\Rightarrow 4\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 8\frac{dy}{dt} + (1-\lambda)y = 0$$

$$\Rightarrow 4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + (1-\lambda)y = 0$$

$$\Rightarrow 4\Delta^2 y + 4\Delta y + (1-\lambda)y = 0$$

where $\Delta^2 = d^2/dt^2$ and $\Delta = d/dt$

$$\Rightarrow (4\Delta^2 + 4\Delta + (1-\lambda))\, y = 0$$

The characteristic equation is

$$4\Delta^2 + 4\Delta + (1-\lambda) = 0$$

which is quadratic in Δ so by Quadractic formula

$$\Delta = \frac{-4 \pm \sqrt{16 - 16(1-\lambda)}}{8}$$

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$$= \frac{-4 \pm 4\sqrt{1-(1-\lambda)}}{8} = \frac{-1 \pm \sqrt{\lambda}}{2}$$

$$\Rightarrow \Delta = -\frac{1}{2} \pm \frac{\sqrt{\lambda}}{2} \quad \text{---------- (2)}$$

There arise three cases for λ.

(i) $\lambda = 0$ (ii) $\lambda > 0$ (iii) $\lambda < 0$.

Since first two cases give trivial solution, therefore we take 3rd case. i.e. $\lambda < 0$.

Take $\lambda = -\mu$ where $\mu > 0$.

Then, (2) $\Rightarrow \Delta = -\frac{1}{2} \pm \frac{i\sqrt{\mu}}{2}$

The general solution is

$$y(t) = \left[C_1 \cos \frac{\sqrt{\mu}}{2} t + C_2 \sin \frac{\sqrt{\mu}}{2} t\right] e^{-\frac{1}{2}t}$$

or

$$y(x) = \left[C_1 \cos \frac{\sqrt{\mu}}{2} \ln(2x+3) + C_2 \sin \frac{\sqrt{\mu}}{2} \ln(2x+3)\right] \frac{1}{\sqrt{2x+3}}$$

as $e^t = 2x+3$

$e^{-t/2} = (2x+3)^{-1/2}$

Applying end point conditions, we get

$$y(0) = 0 \Rightarrow \left[C_1 \cos \frac{\sqrt{\mu}}{2} \ln 3 + C_2 \sin \frac{\sqrt{\mu}}{2} \ln 3\right] \frac{1}{\sqrt{3}} = 0$$

$$\Rightarrow C_1 \cos \frac{\sqrt{\mu}}{2} \ln 3 + C_2 \sin \frac{\sqrt{\mu}}{2} \ln 3 = 0 \quad \text{---------- (3)}$$

$y(3) = 0$

$$\Rightarrow \left[C_1 \cos \frac{\sqrt{\mu}}{2} \ln(9) + C_2 \sin \frac{\sqrt{\mu}}{2} \ln 9\right] \frac{1}{\sqrt{9}} = 0$$

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$\therefore \ln 9 = \ln 3^2 = 2\ln 3$

$$\Rightarrow C_1 \cos\frac{\sqrt{\mu}}{2}\cdot 2\ln 3 + C_2 \sin\frac{\sqrt{\mu}}{2}\cdot 2\ln 3 = 0$$

$$\Rightarrow C_1 \cos\sqrt{\mu}\ln 3 + C_2 \sin\sqrt{\mu}\ln 3 = 0 \quad \text{——(4)}$$

For non trivial solution, from (3) and (4), we must have

$$\begin{vmatrix} \cos\frac{\sqrt{\mu}}{2}\ln 3 & \sin\frac{\sqrt{\mu}}{2}\ln 3 \\ \cos\sqrt{\mu}\ln 3 & \sin\sqrt{\mu}\ln 3 \end{vmatrix} = 0$$

$$\Rightarrow \cos\frac{\sqrt{\mu}}{2}\ln 3 \sin\sqrt{\mu}\ln 3 - \sin\frac{\sqrt{\mu}}{2}\ln 3 \cos\sqrt{\mu}\ln 3 = 0$$

or

$$\Rightarrow \sin\sqrt{\mu}\ln 3 \cos\frac{\sqrt{\mu}}{2}\ln 3 - \cos\sqrt{\mu}\ln 3 \sin\frac{\sqrt{\mu}}{2}\ln 3 = 0$$

$$\Rightarrow \sin\left[\sqrt{\mu}\ln 3 - \frac{\sqrt{\mu}}{2}\ln 3\right] = 0$$

$$\Rightarrow \sin\left[\frac{\sqrt{\mu}}{2}\ln 3\right] = 0$$

$$\Rightarrow \frac{\sqrt{\mu}}{2}\ln 3 = n\pi \quad \text{where } n = \pm 1, \pm 2, \ldots$$

(for $n=0$ we obtain trivial sol.).

$$\Rightarrow \sqrt{\mu} = \frac{2n\pi}{\ln 3}$$

$$\Rightarrow \mu = \frac{4n^2\pi^2}{(\ln 3)^2}; \quad n = 1, 2, \ldots$$

So eigenvalues are

$$\lambda_n = -\frac{4n^2\pi^2}{(\ln 3)^2}; \quad n = 1, 2, 3, \ldots$$

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Subsituting $\frac{\sqrt{\mu}}{2}\ln 3 = n\pi$ in eq (3), we get

$C_1 \cos n\pi + C_2 \sin n\pi = 0$

$\Rightarrow \boxed{C_1 = 0}$

Hence $y_n(x) = \frac{C_n}{\sqrt{2x+3}} \sin\left[\frac{n\pi(\ln(2x+3))}{\ln 3}\right]$; $n = 1, 2 \ldots$

where C_n are constants which can be determined as follows: $\because \int_0^3 y^2 dx = 1$

$$\int_0^3 \frac{C_n^2}{2x+3} \sin^2\left[\frac{n\pi \ln(2x+3)}{\ln 3}\right] dx = 1$$

$$\Rightarrow \frac{C_n^2}{2} \int_0^3 \frac{1}{2x+3}\left[1 - \cos\left(\frac{2n\pi \ln(2x+3)}{\ln 3}\right)\right] dx = 1$$

$$\Rightarrow \frac{C_n^2}{2}\left[\frac{1}{2}\int_0^3 \frac{2dx}{2x+3}dx - \frac{\ln 3}{4n\pi}\int_0^3 \cos\left(\frac{2n\pi \ln(2x+3)}{\ln 3}\right)\frac{4n\pi\, dx}{\ln 3(2x+3)}\right]$$

$$\Rightarrow \frac{C_n^2}{2}\left[\frac{1}{2}\ln(2x+3)\Big|_0^3 - \frac{\ln 3}{4n\pi}\sin\left(\frac{2n\pi \ln(2x+3)}{\ln 3}\right)\Big|_0^3\right] = 1$$

$$\Rightarrow \frac{C_n^2}{2}\left[\frac{1}{2}\ln 9 - \frac{1}{2}\ln 3 - \frac{\ln 3}{4n\pi}\sin\left(\frac{2n\pi \ln 9}{\ln 3}\right)^{0} + \frac{\ln 3}{4n\pi}\sin\left(\frac{2n\pi \ln 3}{\ln 3}\right)^{0}\right] = 1$$

$$\Rightarrow \frac{C_n^2}{2}\left[\ln 3 - \frac{1}{2}\ln 3\right] = 1 \Rightarrow \frac{C_n^2}{2}\left(\frac{1}{2}\ln 3\right) = 1$$

$$\Rightarrow C_n^2 = 4/\ln 3$$

$$\Rightarrow C_n = \pm 2/\sqrt{\ln 3}$$

Awan
10-2-19

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Geodesic Problems.

A geodesic is the curve of shortest length joining two points in space.

Geodesic problems are similar to variational problems with constraints but sometimes can be reduced to problems without constraints.

Example 1.

Find the curve of shortest length b/w two given points in a plane, using polar coordinates (r, θ).

Solution.

The arc length ds is given by

$$ds = \sqrt{dr^2 + r^2 d\theta^2} = \sqrt{r'^2 + r^2}\, d\theta$$

where $r' = dr/d\theta$.

Thus we have to minimize

$$\int_A^B ds = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + r'^2}\, d\theta.$$

Subject to the constraints $r(\theta_1) = \text{Constt}$ and $r(\theta_2) = \text{Constant}$.

Here $F(r, \theta, r') = \sqrt{r^2 + r'^2}$

So, the corresponding Eules-Lagrange equation is

$$F - r'\left(\frac{\partial F}{\partial r'}\right) = \text{Constt}.$$

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$$\Rightarrow \frac{\sqrt{r^2+r'^2}\,r - r'\cdot r'}{\sqrt{r^2+r'^2}} = C_1$$

$$\Rightarrow \frac{r^2+r'^2-r'^2}{\sqrt{r^2+r'^2}} = C_1$$

$$\Rightarrow \frac{r^2}{\sqrt{r^2+r'^2}} = C_1$$

$$\Rightarrow r^2 = C_1\sqrt{r^2+r'^2}$$

$$\Rightarrow r^4 = C_1^2(r^2+r'^2)$$

$$\Rightarrow r^4 = C_1^2 r^2 + C_1^2 r'^2$$

$$\Rightarrow C_1^2 r'^2 = r^4 - C_1^2 r^2$$

$$\Rightarrow r'^2 = \frac{r^4 - C_1^2 r^2}{C_1^2}$$

$$\Rightarrow r' = \sqrt{\frac{r^4 - C_1^2 r^2}{C_1^2}}$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{\sqrt{r^4 - C_1^2 r^2}}{C_1} = \frac{r\sqrt{r^2 - C_1^2}}{C_1}$$

$$\Rightarrow \int \frac{C_1}{r\sqrt{r^2-C_1^2}}\,dr = \int d\theta$$ separating variables

$$\Rightarrow \frac{C_1}{C_1}\sec^{-1}(r/C_1) = \theta + C_2$$

$$\Rightarrow \sec^{-1}\frac{r}{C_1} = \theta + C_2$$

$$\Rightarrow \frac{r}{C_1} = \sec(\theta + C_2)$$

$$\Rightarrow C_1 = r\cos(\theta + C_2)$$

$$= r\{\cos\theta\cos C_2 - \sin\theta\sin C_2\}$$

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$$= (r\cos\theta)\cos c_2 - (r\sin\theta)\sin c_2$$

$$= x\cos c_2 - y\sin c_2$$

$$\Rightarrow c_1 = (\cos c_2)x - (\sin c_1)y$$

which is lineal equation in x, y of the form $\alpha x + \beta y + \gamma = 0$ and represents a straight line.

Example 2. 2015/A

To find the curve of shortest length on the surface of a sphere.

Solution.

Let A and B be two points on the sphere S. we have to minimize the integral

$$I = \int_A^B ds = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

Subject to the constraints $x^2 + y^2 + z^2 = a^2$ where 'a' is the radius of the sphere. Let us use the spherical coordinates (r, θ, φ).

Let (a, θ, φ) and $(a, \theta + d\theta, \varphi + d\varphi)$ be the coordinates of two neighbouring points on the curve through A and B and lying on the sphere.

Then

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$$I = \int_A^B \sqrt{ds^2 + s^2(d\theta)^2 + s^2\sin^2\theta\,(d\varphi)^2}$$

In the case of sphere $s = a \Rightarrow ds = 0$.

So $$I = \int_A^B \sqrt{a^2(d\theta^2) + a^2\sin^2\theta\,(d\varphi)^2}$$

$$= a\int_A^B \sqrt{d\theta^2 + \sin^2\theta\,(d\varphi)^2}$$

$$= a\int_A^B \sqrt{1 + \sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2}\,d\theta$$

$$= a\int_A^B \sqrt{1 + \sin^2\theta\,(\varphi')^2}\,d\theta$$

where $\varphi' = d\varphi/d\theta$

The required curve satisfies the Euler-Lagrange Equation.

$$\frac{\partial F}{\partial \varphi} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial \varphi'}\right) = 0$$

Here $F = \sqrt{1 + \sin^2\theta(\varphi')^2}$

So, $$0 - \frac{d}{d\theta}\left(\frac{1}{2}\{(1+\sin^2\theta\varphi'^2)^{-1/2}(2\varphi'\sin^2\theta)\}\right) = 0$$

$$\Rightarrow -\frac{d}{d\theta}\left(\frac{\varphi'\sin^2\theta}{\sqrt{1+\sin^2\theta\,\varphi'^2}}\right) = 0$$

$$\Rightarrow \frac{\sin^2\theta\,\varphi'}{\sqrt{1+\sin^2\theta\,\varphi'^2}} = G$$

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Squaring on both sides, we get

$$\frac{\sin^4\theta\,\phi'^2}{1+\sin^2\theta\,\phi'^2} = C_1^2$$

$$\Rightarrow \sin^4\theta\,\phi'^2 = C_1^2(1+\sin^2\theta\,\phi'^2)$$

$$\Rightarrow \sin^4\theta\,\phi'^2 = C_1^2 + C_1^2\sin^2\theta\,\phi'^2$$

$$\Rightarrow \sin^4\theta\,\phi'^2 - C_1^2\sin^2\theta\,\phi'^2 = C_1^2$$

$$\Rightarrow \sin^2\theta\,\phi'^2(\sin^2\theta - C_1^2) = C_1^2$$

$$\Rightarrow \phi'^2 = \frac{C_1^2}{\sin^2\theta(\sin^2\theta - C_1^2)}$$

$$\Rightarrow \phi' = \frac{C_1}{\sin\theta\sqrt{\sin^2\theta - C_1^2}}$$

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{C_1}{\sin\theta\sqrt{\sin^2\theta - C_1^2}}$$

$$\Rightarrow \int d\phi = C_1\int \frac{d\theta}{\sin\theta\sqrt{\sin^2\theta - C_1^2}}$$

separating variables

$$= C_1\int \frac{d\theta}{\sin^2\theta\sqrt{1 - C_1^2/\sin^2\theta}}$$

$$\Rightarrow \phi = C_1\int \frac{\operatorname{cosec}^2\theta\,d\theta}{\sqrt{1 - C_1^2\operatorname{cosec}^2\theta}} + C_2$$

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$$\Rightarrow \varphi = C_1 \int \frac{\csc^2 d\theta}{\sqrt{1 - C_1^2(1+\cot^2\theta)}} + C_2$$

Let $\cot\theta = u$ then $-\operatorname{cosec}^2\theta\, d\theta = du$

$$\Rightarrow \varphi = C_1 \int \frac{-du}{\sqrt{1 - C_1^2(1+u^2)}} + C_2$$

$$= -C_1 \int \frac{du}{\sqrt{1 - C_1^2 - C_1^2 u^2}} + C_2$$

$$= -C_1 \int \frac{du}{\sqrt{(1 - C_1^2) - C_1^2 u^2}} + C_2$$

$$= -C_1 \int \frac{du}{C_1\sqrt{\frac{(1-C_1^2)}{C_1^2} - u^2}} + C_2$$

Let $\frac{1 - C_1^2}{C_1^2} = C_3^2$.

Then $\varphi = -\int \frac{du}{\sqrt{C_3^2 - u^2}} + C_2$

$$= \int \frac{-1}{\sqrt{C_3^2 - u^2}}\, du + C_2$$

$$= \cos^{-1}\left(\frac{u}{C_3}\right) + C_2$$

$$= \cos^{-1}\left(\frac{\cot\theta}{C_3}\right) + C_2$$

$$\Rightarrow \varphi - C_2 = \cos^{-1}\left(\frac{\cot\theta}{C_3}\right)$$

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$$\Rightarrow \frac{\cot\theta}{C_3} = \cos(\phi - C_2)$$

$$= \cos\phi\cos C_2 + \sin\phi\sin C_2.$$

$$\Rightarrow \cot\theta = C_3\cos\phi\cos C_2 + C_3\sin\phi\sin C_2$$

$$\Rightarrow \frac{\cos\theta}{\sin\theta} = C_4\cos\phi + C_5\sin\phi$$

where $C_4 = C_3\cos C_2$; $C_5 = C_3\sin C_2$ are new constants

$$\Rightarrow \cos\theta = C_4\sin\theta\cos\phi + C_5\sin\theta\sin\phi$$

$$\Rightarrow a\cos\theta = a(C_4\sin\theta\cos\phi + C_5\sin\theta\sin\phi)$$

$$\Rightarrow a\cos\theta = C_4(a\sin\theta)\cos\phi + C_5(a\sin\theta\sin\phi)$$

$$\Rightarrow z = C_4x + C_5y.$$

which is the equation of plane through the centre of sphere.

Hence the curve of shortest length joining A and B is the arc of the great circle through A and B.

Great Circle.

A circle whose plane passes through the centre of a sphere is called great circle.

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Example 3.

Find the geodesic curve for the cylinder $x^2+y^2=a^2$.

Solution.

In cylinderical coordinates, we have

$x = a\cos\theta$, $y = a\sin\theta$, $z = z$.

We have to minimize

$$l = \int_A^B ds = \int_A^B \sqrt{a^2(d\theta)^2 + dz^2}$$

$$= \int_{\theta_1}^{\theta_2} \sqrt{a^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta$$

$$= \int_{\theta_1}^{\theta_2} \sqrt{a^2 + z'^2}\, d\theta$$

where $z' = dz/d\theta$.

Subject to no constraint.

Here $F = \sqrt{a^2+z'^2}$. The Euler-Lagrange equation is

$$\frac{\partial F}{\partial z} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial z'}\right) = 0.$$

$$\Rightarrow 0 - \frac{d}{d\theta}\left(\frac{z'}{\sqrt{a^2+z'^2}}\right) = 0$$

$$\Rightarrow \frac{z'}{\sqrt{a^2+z'^2}} = C \Rightarrow z'^2 = C^2(a^2+z'^2)$$

$$\Rightarrow z'^2 - C^2z'^2 = C^2a^2$$

$$\Rightarrow z'^2(1-C^2) = C^2a^2 \Rightarrow z' = Ca/\sqrt{1-C^2}$$

$$\frac{dz}{d\theta} = \frac{Ca}{\sqrt{1-C^2}} \Rightarrow \int dz = \frac{Ca}{\sqrt{1-C^2}} \int d\theta$$

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$\Rightarrow z = \frac{ca}{\sqrt{1-c^2}}\theta + c_1 = c_2\theta + c_1$ where $c_2 = \frac{ca}{\sqrt{1-c^2}}$

$\Rightarrow \theta = \frac{z-c_1}{c_2} \Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{z-c_1}{c_2}$

$\Rightarrow y/x = \tan\left(\frac{z-c_1}{c_2}\right)$.

Example 4.

Find the shortest distance between the points $A(1,-1,0)$ and $B(2,1,-1)$ in the plane $15x - 7y + z - 22 = 0$.

Solution.

Here we have to minimize

$$l = \int_A^B \sqrt{dx^2 + dy^2 + dz^2}$$

$$= \int_A^B \sqrt{1 + y'^2 + z'^2}\, dx$$

Subject to the constraints that the points A, B be on the plane

$$G = 15x - 7y + z - 22 = 0$$

Here $F = \sqrt{1 + y'^2 + z'^2}$

So $H = F + \lambda G$

$$= \sqrt{1 + y'^2 + z'^2} + \lambda(15x - 7y + z - 22)$$

The corresponding Euler-Lagrange equations are

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \quad \text{——— (1)}$$

and $$\frac{\partial H}{\partial z} - \frac{d}{dx}\left(\frac{\partial H}{\partial z'}\right) = 0 \quad \text{——— (2)}$$

(1) $\Rightarrow$ $$-7\lambda - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0. \quad \text{— (3)}$$

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$$② \Rightarrow \lambda - \frac{d}{dx}\left(\frac{z'}{\sqrt{1+y'^2+z'^2}}\right) = 0.$$

$$\Rightarrow \lambda = \frac{d}{dx}\left(\frac{z'}{\sqrt{1+y'^2+z'^2}}\right) \quad \text{——— } ④$$

Substituting ④ in ③, we get

$$-7\frac{d}{dx}\left(\frac{z'}{\sqrt{1+y'^2+z'^2}}\right) - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2+z'^2}}\right) = 0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{y'+7z'}{\sqrt{1+y'^2+z'^2}}\right) = 0$$

$$\Rightarrow \frac{y'+7z'}{\sqrt{1+y'^2+z'^2}} = c$$

Squaring on both sides, we get

$$(y'+7z')^2 = c^2(1+y'^2+z'^2) \quad \text{——— } ⑤$$

Also, we have equation of plane

$$15x - 7y + z - 22 = 0$$

Diff. w.r.t. x, we get

$$15 - 7y' + z' = 0$$

$$\Rightarrow z' = 7y' - 15 \quad \text{——— } ⑥$$

Substituting ⑥ in ⑤, we get

$$\left[y' + 7(7y' - 15)\right]^2 = c^2\left[1 + y'^2 + (7y' - 15)^2\right]$$

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$$\Rightarrow [y' + 49y' - 105]^2 = C^2 [1 + y'^2 + 49y'^2 + 225 - 210y']$$

$$\Rightarrow (50y' - 105)^2 = C^2 (50y'^2 - 210y' + 226)$$

$$\Rightarrow 2500y'^2 + 11025 - 10500y' = 50C^2 y'^2 - 210C^2 y' + 226C^2$$

$$\Rightarrow (2500 - 50C^2)y'^2 + (210C^2 - 10500)y' + (11025 - 226C^2) = 0.$$

$$\Rightarrow 50(250 - C^2)y'^2 + 210(C^2 - 50)y' + (11025 - 226C^2) = 0$$

which is a quadratic equation in y'.

Since C is an arbitrary constant, we can choose C such that equation has real root. Let α be one such real root, then

$$y' = \alpha$$

$$\Rightarrow y = \alpha x + \beta$$

Applying the given B.Cs, we get

$y(1) = -1 \Rightarrow \alpha + \beta = -1$ ——— (*)

$y(2) = 1 \Rightarrow 2\alpha + \beta = 1$ ——— (**)

(**) − (*) $\Rightarrow \boxed{\alpha = 2}$

(*) $\Rightarrow 2 + \beta = -1 \Rightarrow \boxed{\beta = -3}$

Thus $y = 2x - 3$

For Z, we have

$$Z = 7y - 15x + 22$$

$$= 7(2x - 3) - 15x + 22$$

$$= 14x - 21 - 15x + 22 = -x + 1$$

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The required least distance is

$$l = \int_1^2 \sqrt{1+y'^2+z'^2}\, dx$$

$$= \int_1^2 \sqrt{1+(2)^2+(-1)^2}\, dx = \int_1^2 \sqrt{1+4+1}\, dx$$

$$= \sqrt{6}\, x \Big|_1^2 = \sqrt{6}\,(2-1)$$

$$= \sqrt{6}.$$

Example 5.

Find the shortest distance between the points $A(1,0,-1)$ and $B = (0,-1,1)$ lying on the surface $x+y+z=0$.

Solution.

we have to minimize

$$l = \int_A^B \sqrt{(dx)^2+(dy)^2+(dz)^2}$$

$$= \int_0^1 \sqrt{1+y'^2+z'^2}\, dx.$$

Subject to the constraint that the points A, B be on the surface

$$G = x+y+z = 0.$$

Here $F = \sqrt{1+y'^2+z'^2}$

so $H = F + \lambda G = \sqrt{1+y'^2+z'^2} + \lambda(x+y+z)$

The corresponding Eulel-Lagrange equations are

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$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \quad \text{——— (1)}$$

and $$\frac{\partial H}{\partial z} - \frac{d}{dx}\left(\frac{\partial H}{\partial z'}\right) = 0 \quad \text{——— (2)}$$

$$(1) \Rightarrow \lambda - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2+z'^2}}\right) = 0$$

$$\Rightarrow \lambda = \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2+z'^2}}\right) = 0 \quad \text{— (3)}$$

$$(2) \Rightarrow \lambda - \frac{d}{dx}\left(\frac{z'}{\sqrt{1+y'^2+z'^2}}\right) = 0 \quad \text{——— (4)}$$

(3) in (4)

$$\Rightarrow \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2+z'^2}}\right) - \frac{d}{dx}\left(\frac{z'}{\sqrt{1+y'^2+z'^2}}\right) = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y'-z'}{\sqrt{1+y'^2+z'^2}}\right) = 0$$

$$\Rightarrow \frac{y'-z'}{\sqrt{1+y'^2+z'^2}} = C$$

Taking square, we get

$$(y'-z')^2 = C^2(1+y'^2+z'^2) \quad \text{——— (5)}$$

Also, we have $x+y+z = 0$

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$\Rightarrow 1+y'+z'=0 \Rightarrow z'=-1-y'$ ——⑥

Substituting ⑥ in ⑤, we get

$(y'-(-1-y'))^2 = c^2[1+y'^2+(-1-y')^2]$

$\Rightarrow (2y'+1)^2 = c^2[1+y'^2+1+y'^2+2y']$

$\Rightarrow (2y')^2+1+4y' = c^2(2y'^2+2y'+2)$

$\Rightarrow 4y'^2+4y'+1 = 2c^2y'^2+2c^2y'+2c^2$

$\Rightarrow (4-2c^2)y'^2+(4-2c^2)y'+(1-2c^2)=0$

which is quadratic equation in y'.

Since c is an arbitrary constant, we can choose c such that equation has real root. Let α be one such real root, then

$y' = \alpha$

$\Rightarrow y = \alpha x+\beta.$

Applying the given B.C's, we get

$y(0)=-1 \Rightarrow \boxed{\beta=-1}$

$y(1)=0 \Rightarrow \alpha+\beta=0 \Rightarrow \alpha=-\beta$

$\Rightarrow \boxed{\alpha=1}$

Thus $y = x-1$

For z, we have $z = -x-y$

$\Rightarrow z = -x-x+1 = -2x+1.$

The required distance is,

$l = \int_0^1 \sqrt{1+y'^2+z'^2}\,dx = \int_0^1 \sqrt{1+1+4}\,dx$

$= \sqrt{6}\,x\Big|_0^1 = \sqrt{6}(1-0) = \sqrt{6}.$

Shaul

12-2-19

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Question:- A/2014.

Find the curve joining the points (0, 0) and (1, 0) such that the y coordinate of its centroid is minimum.

Solution

Same as example 1 page 231

only change the condition $y(a)=0$ by $y(1)=0$.

$\Rightarrow C_2 = -1/2$

Thus $y = C_1\left[\cosh\frac{x-1/2}{C_1} - \cosh\left(\frac{1}{2C_1}\right)\right]$

This curve is called catenary.

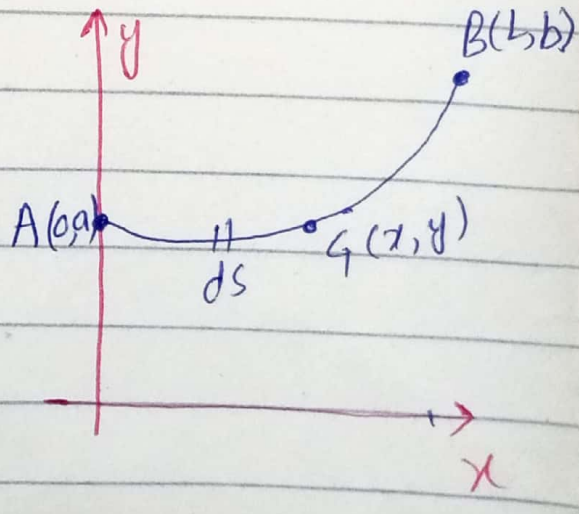
Question. A/2012.

A chain is suspended between two points A(0, a) and B(L, b). Find the shape of the chain in equilibrium.

Solution

Same as example 1 page 231.

only change the conditions $y(0)=0$ and $y(a)=0$ by $y(0)=a$ and $y(L)=b$.



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